On the semantics of proofs in classical sequent calculus.

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Abstract. We discuss the problem of finding non-trivial invariants of non-deterministic, symmetric cut-reduction procedures in the classical sequent calculus. We come to the conclusion that (an enriched version of) the propositional fragment of GS4 – i.e. the one-sided variant of Kleene's context-sharing style sequent system G4, where independent rule applications permute freely – is an ideal framework in which to attack the problem. We show that the graph induced by axiom rules linking dual atom occurrences is preserved under arbitrary rule permutations in the cut-free fragment of GS4. We then refine the notion of axiom-induced graph so as to extend the result to derivations with cuts, and we exploit the invertibility of logical rules to define a global normalisation procedure that preserves the refined axiom-induced graphs, thus yielding a non-trivial invariant of cut-elimination in GS4. Finally, we build upon the result to devise a new proof system for classical propositional logic, where the rule permutations of GS4 reduce to identities.

Keywords: Classical propositional logic · Sequent calculus · Cut-elimination · Invariants of cut-reduction · Proof-identity · Denotational semantics.

1 Introduction

Cut-elimination procedures in classical sequent calculus are notoriously non-deterministic and non-confluent, both in the original formulation by Gentzen and in later reformulations [12, 10, 4, 5, 1, 30]. It is natural to ask whether those instances of non-confluence are superficial in nature, i.e. whether distinct normal forms of the same derivation are in fact correlated in a non-trivial way. A famous counter-example by Lafont [16] purports to show that the answer is negative, that is, any notion of proof equivalence compatible with classical cut-elimination must be a trivial one that identifies all proofs of the same sequent. Specifically, the counter-example involves a derivation of the form

$$\dfrac{\dfrac{\dfrac{\vdots\, P}{\vdash A}}{\vdash A, B}\,\texttt{wk} \qquad \dfrac{\dfrac{\vdots\, Q}{\vdash A}}{\vdash A, \overline{B}}\,\texttt{wk}}{\dfrac{\vdash A, A}{\vdash A}\,\texttt{ctr}}\,\texttt{cut}$$

where P, Q are a pair of arbitrary derivations of the same formula A.[1] Because both cut-formulas are introduced by weakening, there are two ways to eliminate the cut, leading to two potentially very different derivations:

$$\dfrac{\dfrac{\dfrac{\vdots\, P}{\vdash A}}{\vdash A, A}\,\texttt{wk}}{\vdash A}\,\texttt{ctr} \qquad\qquad \dfrac{\dfrac{\dfrac{\vdots\, Q}{\vdash A}}{\vdash A, A}\,\texttt{wk}}{\vdash A}\,\texttt{ctr}$$

Any notion of proof identity that is compatible with the cut-elimination process should then identify all three derivations shown above, leading to the identification of P with Q under the reasonable assumption that the weakening-contraction sequence on A be an irrelevant detour. P and Q, however, were *arbitrary* derivations of A, i.e. potentially not correlated in any way: this has catastrophic consequences, making our hypothetical notion of proof identity trivial and patently unreasonable. For example, the following three proofs would be identified:

$$\dfrac{\dfrac{\vdots}{\vdash A}}{\vdash A \vee B}\,\vee \qquad\qquad \dfrac{\dfrac{\vdots}{\vdash B}}{\vdash A \vee B}\,\vee \qquad\qquad \dfrac{\dfrac{\vdots}{\vdash A, B}}{\vdash A \vee B}\,\vee$$

meaning that it does not matter at all whether a disjunction holds in virtue of the left or of the right disjunct, or by contradiction. In a sense we are identifying not just all proofs of the same thesis, but indeed *all* proofs altogether.

Under such a notion of proof identity, it becomes utterly useless to look for alternative proofs of the same theorem, as they are virtually the same as any other one and there is nothing to be learned from them: a conclusion that plainly contradicts millennia of mathematical practice.

Many solutions have been proposed. We mention among them Krivine's *classical realizability* program [23, 22], which has yielded a fruitful analysis of classical reasoning principles in computational terms; the approaches based on polarised deductive systems, like Girard's calculus LC [13], Parigot's $\lambda\mu$-calculus [28], Danos Joinet and Schellinx' calculi LKT and LKQ [10], and finally the approaches based on fine grained focalization and embeddings of the classical sequent calculus into linear logic [26, 9].

From the point of view of sequent calculus, the common thread to all mentioned solutions is to restrict cut-elimination (and sometimes the logical rules too) in a principled way, breaking its simmetry and therefore solving the non-deterministic cases. Typically the approaches based on λ-calculus correspond to various double-negation translations from classical to intuitionistic logic that preserve provability. While this approach bears many interesting results, it is not entirely satisfactory.

[1] We use the one-sided formulation of sequent calculus in the style of Tait [34] to reduce the number of rules to be treated. Negation is defined as an involution on formulas through De Morgan dualities. Sequents are considered as multisets, i.e. quotiented up to arbitrary permutations of their elements, hence the exchange rule is implicit

On the one hand, every restriction is somewhat arbitrary. Looking back at Lafont's example above, it is hard to see any actual reason to privilege one reduct over the other, and doing so arguably fails to capture the content of the original derivation, which effectively offers two possible ways to prove the same conclusion. On the other hand, the problem is by no means exclusive to sequent calculus. Analogous examples can be constructed in a great variety of classical proof systems, with the notable exception of those who have been accurately tuned to avoid them, and there is even a categorical counterpart due to Joyal. The solutions recalled above sacrifice the inherent symmetries of classical proof systems to the possibility of extracting at least some computational content from them. It is important to underline here that while those symmetries are not strictly needed to characterize classical provability, the fact that classical logic allows them to arise seems to be somehow essential to it.

A long standing open question has been then whether it could possible to work around the non-deterministic reduction steps by natural and non-trivial adjustments of the calculus and/or of cut-reduction steps, without resorting to symmetry-breaking techniques like polarization or embeddings into intuitionistic or linear logic. It is clear that Lafont's example requires a change in the calculus. Two simple solutions have been discussed many times in the literature: the *mix* rule and *non-deterministic sums*.

$$\dfrac{\overset{\vdots\; P}{\vdash \Gamma} \qquad \overset{\vdots\; Q}{\vdash \Delta}}{\vdash \Gamma, \Delta}\,\texttt{mix} \quad \xleftarrow{?} \quad \dfrac{\dfrac{\overset{\vdots\; P}{\vdash \Gamma}}{\vdash \Gamma, A}\,\texttt{wk} \qquad \dfrac{\overset{\vdots\; Q}{\vdash \Delta}}{\vdash \Delta, \overline{A}}\,\texttt{wk}}{\vdash \Gamma, \Delta}\,\texttt{cut} \quad \xrightarrow{?} \quad \dfrac{\dfrac{\overset{\vdots\; P}{\vdash \Gamma}}{\vdash \Gamma, \Delta}\,\texttt{wk} \qquad \dfrac{\overset{\vdots\; Q}{\vdash \Delta}}{\vdash \Gamma, \Delta}\,\texttt{wk}}{\vdash \Gamma, \Delta}\,\oplus$$

While those two rules do not contribute to classical provability, they increase the amount of available proofs [25], accommodating for cut-free proofs that intuitively "provide multiple alternatives."

The non-deterministic sum (on the right) expresses the intuitive idea that the resulting proof might be seen either as P or Q, without the reader being free to choose between the two alternatives. Its use can be extended coherently to all problematic situations, viewing proofs with cuts as the sum of their possible normal forms. While this approach is known to be non-trivial and non-reducible to intuitionistic or linear embeddings (see e.g. [5, 24]), it is also not very deep, as it does not get to the point of what the content of a classical proof actually is; notably, it is not known to yield a canonical representation of classical proofs up to the related notion of proof identity. Moreover, it has no faithful representation on paper, where the choice between P and Q is clearly available to the reader.

On the other hand, the mix rule (on the left) expresses the idea that two proofs are effectively provided in parallel, both equally available to the reader. Informally, this might be viewed as writing down two different proofs of the same theorem one after the other: something that makes sense and might actually happen in mathematics textbooks. Extending the calculus with the mix rule

$$
\dfrac{\dfrac{\begin{array}{c}\vdots\, P\\ \vdash \Gamma, A, B\end{array} \quad \begin{array}{c}\vdots\, Q\\ \vdash \Delta, \overline{A}\end{array}}{\vdash \Gamma, \Delta, B}\,\mathtt{cut} \quad \begin{array}{c}\vdots\, R\\ \vdash \Sigma, \overline{B}\end{array}}{\vdash \Gamma, \Delta, \Sigma}\,\mathtt{cut}
\quad\longleftarrow\quad
\dfrac{\dfrac{\begin{array}{c}\vdots\, P\\ \vdash \Gamma, A, B\end{array}}{\vdash \Gamma, A \vee B}\,\vee \quad \dfrac{\begin{array}{c}\vdots\, Q\\ \vdash \Delta, \overline{A}\end{array} \quad \begin{array}{c}\vdots\, R\\ \vdash \Sigma, \overline{B}\end{array}}{\vdash \Delta, \Sigma, \overline{A} \wedge \overline{B}}\,\wedge}{\vdash \Gamma, \Delta, \Sigma}\,\mathtt{cut}
\quad\longrightarrow\quad
\dfrac{\dfrac{\begin{array}{c}\vdots\, P\\ \vdash \Gamma, A, B\end{array} \quad \begin{array}{c}\vdots\, R\\ \vdash \Sigma, \overline{B}\end{array}}{\vdash \Gamma, A, \Sigma}\,\mathtt{cut} \quad \begin{array}{c}\vdots\, Q\\ \vdash \Delta, \overline{A}\end{array}}{\vdash \Gamma, \Delta, \Sigma}\,\mathtt{cut}
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\dfrac{\begin{array}{c}\vdots\, P\\ \vdash \Gamma\end{array}}{\vdash \Gamma, \Delta}\,\mathtt{wk}
\quad\longleftarrow\quad
\dfrac{\dfrac{\begin{array}{c}\vdots\, P\\ \vdash \Gamma\end{array}}{\vdash \Gamma, A}\,\mathtt{wk} \quad \dfrac{\begin{array}{c}\vdots\, Q\\ \vdash \Delta\end{array}}{\vdash \Delta, \overline{A}}\,\mathtt{wk}}{\vdash \Gamma, \Delta}\,\mathtt{cut}
\quad\longrightarrow\quad
\dfrac{\begin{array}{c}\vdots\, Q\\ \vdash \Delta\end{array}}{\vdash \Gamma, \Delta}\,\mathtt{wk}
$$

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\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\, P\\ \vdash \Gamma, A, A\end{array}}{\vdash \Gamma, A}\,\mathtt{ctr} \quad \dfrac{\dfrac{\begin{array}{c}\vdots\, P\\ \vdash \Gamma, A, A\end{array}}{\vdash \Gamma, A}\,\mathtt{ctr} \quad \begin{array}{c}\vdots\, Q\\ \vdash \Delta, \overline{A}, \overline{A}\end{array}}{\vdash \Gamma, \Delta, \overline{A}}\,\mathtt{cut}}{\vdash \Gamma, \Delta, \Delta}\,\mathtt{cut}}{\vdash \Gamma, \Delta}\,\mathtt{ctr}
\quad\longleftarrow\quad
\dfrac{\dfrac{\begin{array}{c}\vdots\, P\\ \vdash \Gamma, A, A\end{array}}{\vdash \Gamma, A}\,\mathtt{ctr} \quad \dfrac{\begin{array}{c}\vdots\, Q\\ \vdash \Delta, \overline{A}, \overline{A}\end{array}}{\vdash \Delta, \overline{A}}\,\mathtt{ctr}}{\vdash \Gamma, \Delta}\,\mathtt{cut}
\quad\longrightarrow\quad
\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\, P\\ \vdash \Gamma, A, A\end{array} \quad \dfrac{\begin{array}{c}\vdots\, Q\\ \vdash \Delta, \overline{A}, \overline{A}\end{array}}{\vdash \Delta, \overline{A}}\,\mathtt{ctr}}{\vdash \Gamma, \Delta, A}\,\mathtt{cut} \quad \dfrac{\begin{array}{c}\vdots\, Q\\ \vdash \Delta, \overline{A}, \overline{A}\end{array}}{\vdash \Delta, \overline{A}}\,\mathtt{ctr}}{\vdash \Gamma, \Delta, \Delta}\,\mathtt{cut}}{\vdash \Gamma, \Delta}\,\mathtt{ctr}
$$

Fig. 1: Non-deterministic cut-reduction steps in the classical sequent calculus; also called *logical* and *structural dilemmas* in [9].

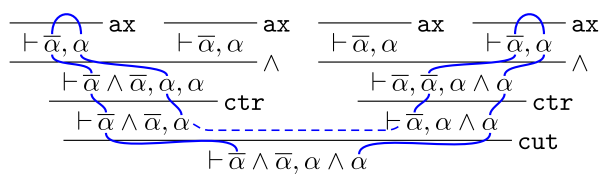
effectively solves the weakening-weakening problem.[2] The question becomes then how to handle the other two non-deterministic reduction steps (fig. 1).

We start from an idea which is well-established in proof-theory, i.e. that the way axioms link together the subformulas of the conclusion is somehow essential to the content of a proof. Such an idea underlines, e.g., the theory of proof-nets [15] and the research program known as *Geometry of Interaction* [14], but has also been widely explored – for various purposes and in many different ways – in the classical setting, among others by Andrews [2, 3], Statman [32], Buss and Carbone [6, 7], Lamarche and Straßburger [25, 33], Hughes [19, 20], Guglielmi and Gundersen [17].

The approach of Andrews, Lamarche and Straßburger in particular is of interest here: it consists in extracting graphs from each derivation, whose vertices are the atomic formula occurrences in the conclusion and whose edges join those dual occurrences that are related by some axiom. Andrews first considered graphs of this kind (which he called *matings*) in [2], and showed in [3] how to reconstruct natural deduction proofs from them. Lamarche and Straßburger used them in [25] to develop a system of proof-nets for classical logic (which they called $\mathbb{B}$-*nets*), and provided a correctness criterion and a sequentialization theorem, as well as a confluent and terminating cut-elimination procedure on nets which corresponds essentially to a composition of graphs by contraction of alternating paths, in the style of the Geometry of Interaction.

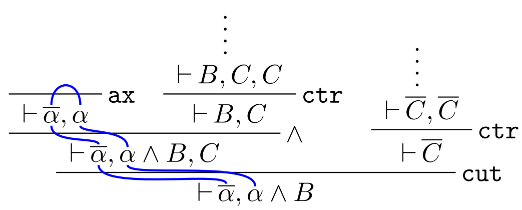
[2] Note however that there is a subtle technical issue to be handled: because logical rules might hide applications of weakenings on their subformulas, weakening cuts cannot be eliminated directly; instead, all weakenings must first be reduced to atomic form, something which is well-known to be possible in classical logic.

Having such a procedure is of the essence when extracting graphs from derivations with cuts. One has to trace the history of all atomic occurrences through the derivation, then combine the traces with axioms and cuts in an appropriate way to obtain the resulting graph. As an example consider the following derivation:

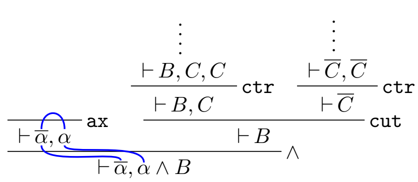


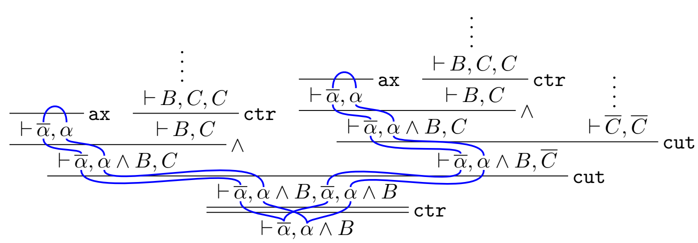
We select an atomic formula occurrence in the conclusion and start tracing its history up through the derivation; when we reach an axiom we move to the linked dual occurrence and start traveling down; when we reach a cut we move to the corresponding dual occurrence in the other premiss and start moving up again; we continue doing so until we get back to the conclusion: the graph shall then contain an edge linking the initial and final points of the resulting path.

The key idea is however not just to track the axioms, but also to *avoid counting them*. To understand why this approach might hold promise for a better treatment of the contraction case, consider the following example:



Notice that there is exactly one axiom rule linking the two dual occurences of α in the conclusion. There are two possible ways to reduce the cut:

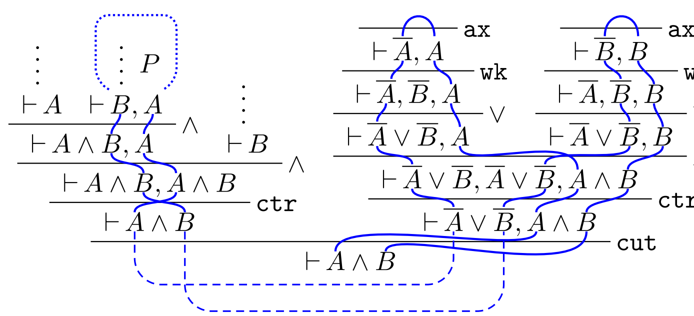




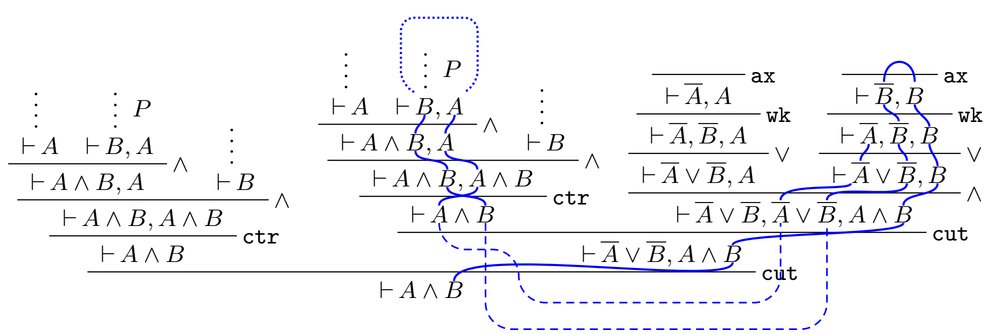
On the left, the cut has been commuted with the conjunction rule. Axioms in the new derivation have not changed at all. On the right, the cut has been reduced into two cuts of lower complexity, and the left subderivation has been duplicated. There are now clearly more axioms than before, *but they link the exact same subformulas.* For example, two axioms now link the dual occurrences of α in the conclusion, but no new link has been created.

The question is then whether axiom-induced graphs, being insensitive to changes in the number of axioms in a derivation, are indeed preserved by all non-deterministic cut-reduction steps. Lamarche and Straßburger were motivated by results obtained in the family of formalisms known as *Deep Inference*, and while they provided an interpretation of the sequent calculus into $\mathbb{B} - nets$, as far as we know they didn't investigate in detail the behaviour of the interpretation under cut-elimination. Führmann and Pym have proven in [11] – as a general theorem for a whole class of interpretations of the classical sequent calculus – that the axiom-induced graphs cannot gain edges under cut-reduction, i.e. that our intuition that no axioms are created is indeed correct. They also show that the graphs are preserved by the two logical cut-reduction steps, and it is possible to prove that they are also invariant under atomic weakening reductions when the non-deterministic case is solved by the mix rule.

Unfortunately, it turns out they are *not* preserved by all cut-reduction steps involving contraction, as there are cases where some paths disappear. The prototypical counter-example, also due to Führmann and Pym [11], looks like this:



Observe that, under the assumption that the subderivation P have a path connecting B with A (represented here as a dotted line), there is in the axiom graph of the complete derivation a path connecting (some subformula of) A with (some subformula of) B in the conclusion. When the cut is reduced by duplicating the right subderivation and commuting it up the two conjunctions, nothing bad happens. However, if the cut is reduced by duplicating the left subderivation, the path traced above is lost:



We show for clarity only the surviving part of the path that is connected to B.

Note how the new path must end at the weakening rule. The old path depended critically on the ability to pass through both premises of the contraction rule; now that the contraction rule has vanished and we have two independent cuts it becomes impossible to construct the same path.

There is however a better way to look at the same problem. Notice how the following two derivations are associated to the same axiom graph:

$$\dfrac{}{\vdash \overline{A}\vee\overline{B}, A\wedge B}\,\mathtt{ax} \qquad\qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \overline{A}, A}\,\mathtt{ax}}{\vdash \overline{A},\overline{B},A}\,\mathtt{wk}}{\vdash \overline{A}\vee\overline{B},A}\,\vee \quad \dfrac{\dfrac{\dfrac{}{\vdash \overline{B},B}\,\mathtt{ax}}{\vdash \overline{A},\overline{B},B}\,\mathtt{wk}}{\vdash \overline{A}\vee\overline{B},B}\,\vee}{\vdash \overline{A}\vee\overline{B},\overline{A}\vee\overline{B},A\wedge B}\,\wedge}{\vdash \overline{A}\vee\overline{B},A\wedge B}\,\mathtt{ctr}$$

The peculiarity of this graph is that it is behaves as an identity w.r.t. composition, i.e. whatever derivation P we may cut against one of the two derivations above, the resulting graph will be precisely that of P. This is not in the least surprising in the case of the derivation of the left, which is just an axiom hence an identity w.r.t. cut-elimination. The situation is different for the derivation on the right, which is not in general an identity w.r.t. cut-elimination.

The principle expressed by the derivation on the right is the *syntactic invertibility of conjunctions*, i.e. the fact that any derivation P of the sequent $\vdash \Gamma, A\wedge B$ can be turned into a derivation P' of the same sequent whose last rule (up to some auxiliary contraction on the context) introduces the conjunction $A\wedge B$. One way to obtain that result is precisely to cut P with the derivation on the right, then eliminate the cut by duplicating P and commuting its copies up.

The difficulty with this formulation of the classical sequent calculus is that it contains derivations of conjunctions whose axiom graph cannot be expressed by a derivation ending with the conjunction rule. This is precisely the case for the left subderivation of the counter-example described above. The result of our analysis suggests the conjecture that drives our methodological approach: either axiom graphs are the natural semantics of a proof system where the invertibility of logical rules is a fundamental property, and essentially an identity, or at least we may hope to solve the problem by moving to such a system and refining the notion of axiom graph until it becomes invariant under inversion of logical rules.

The sequent calculus `GS4` (displayed in fig. 2), both in full and in its cut-free fragment, is precisely such a system. It is the one-sided presentation of Kleene's context-sharing style calculus `G4` [21, 18, 35], where independent rule applications permute freely. It is known that, when the axioms of `GS4` are restricted to atomic conclusions, the set of axioms of any derivation is invariant under arbitrary permutations of logical rules [29, 27]. The calculus admits an elegant proof of completeness and enjoys a cut-elimination procedure [30].

One difficulty in adopting `GS4` for our investigation is that every provable sequent has a unique derivation up to permutations of logical rules. This is due to the fact that weakenings are absorbed into the axioms and sequents are identified

$$\frac{}{\vdash \Gamma, \alpha, \overline{\alpha}}\,\texttt{ax}\ (\Gamma \text{ atomic}) \qquad \frac{\vdash \Gamma, A \quad \vdash \Gamma, \overline{A}}{\vdash \Gamma}\,\texttt{cut}$$

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}\,\vee \qquad \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\,\wedge$$

Fig. 2: The sequent calculus GS4.

up to arbitrary permutations of their elements, hence we lose in general the ability to tell which pairs of atomic formulas are linked by the axiom:

$$\frac{}{\vdash \beta, \overline{\beta}, \alpha, \overline{\alpha}}\,\texttt{ax} \quad = \quad \frac{}{\vdash \alpha, \overline{\alpha}, \beta, \overline{\beta}}\,\texttt{ax}$$

In order to restore that ability we perform two steps (section 2):

- we replace usual formulas with *named formulas* (definition 1), i.e. formulas where each atomic formula occurrence is assigned a distinct name:

  $$\alpha^x \vee (\beta^y \wedge \alpha^z).$$

  It is a tedious but unfortunately necessary technical detail, as it is the only way keep track of the history of each occurrence through the derivation;[3]
- we enrich the calculus with a *deterministic axiom* and a *superposition rule*:

  $$\frac{}{\vdash \Gamma, \alpha^x, \overline{\alpha}^y}\,\texttt{ax}_{\{\alpha^x, \overline{\alpha}^y\}} \qquad \frac{\vdash \Gamma \quad \vdash \Gamma}{\vdash \Gamma}\,\sqcup$$

We additionally allow the use of non-atomic axioms, which are interesting in that they greatly reduce the size of derivations; this makes reasoning about the calculus a bit more complicated, but it provides us with a nice proof of derivability of the contraction rule (section 3.1).

Deterministic axioms allow us to recover the standard form that axioms have in calculi with explicit structural rules without losing the benefits of axioms with embedded weakenings. They essentially provide a way to tell which formulas in the context come from a weakening, and which ones do not. Observe that this is not a trivial addition to GS4: the complexity of checking the correctness of standard axiom rule applications ranges from linear to quadratic in the size of the context, while for deterministic axioms it ranges from sub-linear to linear.

The superposition rule is an adaptation of the mix-rule to the context-sharing framework. The shape of the rule is reminiscent of the non-deterministic sums discussed earlier, but its behaviour under cut-elimination is, as we shall see, closer

[3] The alternative is the sequents-as-lists approach, which requires no modification whatsoever to the calculus, but makes reasoning on proof transformations exceedingly complicated because of the need to insert exchange rules everywhere.

to that of the mix-rule. We choose to adopt a non-standard name for the rule to remark this fact and avoid confusion.

In section 3 we recall the main results about the invertibility of logical rules in GS4, and introduce some related notation. We slightly depart from standard terminology in that we speak of *inversion* when recovering derivations of the premises of a logical rule, while we speak of *isolation* when permuting rule applications to the bottom of the derivation. The two properties are provably equivalent, hence they often receive the same name. In our case, however, we needed a way to distinguish the two operations.

We are then able to formalize the axiom graph construction on the enriched calculus (section 4, definition 9) and show the remarkable fact that axiom graphs are preserved under inversion of logical rules in the cut-free fragment (proposition 2). The full calculus on the other hand fails to satisfy the same property; two counter-examples are provided in figs. 4 and 5 and discussed in section 4.2.

This motivates us to seek a refinement of the axiom graph construction. The main problem identified in section 4.2 is that the composition operator for axiom graphs might join edges which occur in incompatible branches of the derivation, i.e. branches that prove distinct conjuncts of some conjunction occurring in the conclusion. Thus a path is obtained which provably cannot exist in a cut-free derivation.

Our solution is to enrich axiom graphs with labels tracking for each edge the branch it came from: we are then able to define a branch-sensitive composition operator (definition 13). We develop the refinement in section 5, then show in section 6.1 that the new construction is invariant under inversion of logical rules in the full fragment of the calculus (theorem 2).

Unfortunately, the refined axiom graphs are no longer preserved by the standard logical cut-reduction steps (as defined e.g. in [30]). The reasons are subtle, we discuss them briefly along with possible solutions in section 8.

We then put our hand to developing a new normalization procedure that preserves the refined axiom graphs. Naturally, invertibility is at the core of the procedure (described in the proof of theorem 3), which exploits it to permute all cuts up the derivation until they are reduced to *atomic contexts*, i.e. they are of the form

$$\frac{\vdash \Gamma, A \quad \vdash \Gamma, \overline{A}}{\vdash \Gamma}\,\texttt{cut}$$

where Γ contains only atomic formulas, while A may be arbitrarily complex. The lack of a proper cut-reduction procedure (section 8) forces us to proceed from this point with a normalisation-by-evaluation argument, i.e. we show that whenever the graph of a derivation is not empty, there is a cut-free derivation of the same conclusion and associated to the same graph.

Such a result was essentially already available in [25]. We manage to restrict the need for this kind of argument to the significantly simpler case of cuts with atomic context between cut-free derivations. Nonetheless the main lemma (lemma 14, a kind of graph-based cut-admissibility result) still requires a very complex proof (provided in appendix G) which a proper cut-reduction procedure, if available, would simplify greatly.

Finally, we provide in section 7 a direct characterisation of the class of axiom graphs that come from `GS4` derivations. The resulting condition, which we call *totality*, is analogous to the correctness criterion of [25] and provides a kind of sequentialization theorem theorem 4. We exploit this fact to devise a classical proof system (which we call `BLG`) optimized for invertibility, in the sense that the inversion procedures become immediate and the isolation procedures are just identities. Proofs in `BLG` have the following shape:

$$\frac{\alpha^x, \beta^z, \overline{\alpha}^v, \gamma^w \quad \alpha^x, \overline{\gamma}^u, \overline{\alpha}^v, \gamma^w \quad \overline{\beta}^y, \beta^z, \overline{\alpha}^v, \gamma^w \quad \overline{\beta}^y, \overline{\gamma}^u, \overline{\alpha}^v, \gamma^w}{\vdash \alpha^x \wedge \overline{\beta}^y, \beta^z \wedge \overline{\gamma}^u, \overline{\alpha}^v \vee \gamma^w}$$

i.e. a set of atomic decompositions of the conclusion together with a graph providing the axiom links. All `GS4` rules, including the cut-rule, are admissible in `BLG`.

It is important to stress how `BLG` is different from the $\mathbb{B}$-nets of [25], apart from disallowing some proofs. The question of whether a certain formalism provides a proof system for propositional classical logic is delicate. Because propositional tautologies are decidable in exponential time in the size of the formula, it has been argued (e.g. in [8, 20]) that correctness criteria for proofs should have significantly lower complexity. In particular, Cook and Reckhow argue in [8] that, as a minimal requirement, proofs should be checkable in polynomial time for some adequate notion of proof size.

$\mathbb{B}$-nets notoriously failed to be a proof system in this sense: the only known correctness criterion is exponential in the size of the proof object, which is itself polynomially bounded by the size of the conclusion; moreover, Das showed that the existence of a polynomial time correctness criterion for Andrews' matings [2] or for $\mathbb{B}$-nets would imply $\mathbf{NP} = \mathbf{coNP}$.

The situation is different for `BLG`, where we are able to argue (quite informally) that correctness is checkable in polynomial time in the size of proof objects (proposition 5). The idea is that a correctness check amounts to constructing a `GS4` derivation of the same conclusion, then check that the decomposition provided by the `BLG` proof object matches the one obtained through `GS4`. Crucially, it is possible to construct the derivation one branch at a time and thus check the correctness of the decomposition incrementally: we are then able to show that the number of required steps is polynomially bounded by the size of the `BLG` proof object.

2 Tracking atom occurrences

Let us fix a *countably infinite* set $\mathcal{N}$ of *names* and an arbitrary set $\mathcal{A}$ of *propositional atoms*, together with a fixpoint-free involution

$$\overline{(\cdot)} : \mathcal{A} \to \mathcal{A}$$

i.e. a map such that $\alpha \neq \overline{\alpha}$ and $\overline{\overline{\alpha}} = \alpha$ for all atoms $\alpha \in \mathcal{A}$: this means that atoms come in pairs symmetrically related by the involution. We use letters $x, y, z, \ldots$ to range over $\mathcal{N}$ and greek letters $\alpha, \beta, \gamma, \ldots$ to range over $\mathcal{A}$.

Definition 1 (Named formulas). *The set $\mathcal{F}$ of* classical propositional formulas *is defined by the grammar*

$$F, G ::= \alpha \mid F \vee G \mid F \wedge G$$

where $\alpha \in \mathcal{A}$. Named formulas *are formulas where every occurrence of a propositional atom is assigned a name from the set $\mathcal{N}$. Formally, we define the set $\mathcal{F}^{\mathcal{N}}$ of* formulas with names in $\mathcal{N}$ *by the grammar*

$$A, B ::= \alpha^x \mid A \vee B \mid A \wedge B$$

where $\alpha \in \mathcal{A}$ and $x \in \mathcal{N}$. Call (possibly named) formulas of the form α (resp. α^x) atomic*. Each named formula $A \in \mathcal{F}^{\mathcal{N}}$ is naturally associated to the anonymous formula $|A| \in \mathcal{F}$ obtained by forgetting all names. Write $A \equiv B$ iff $|A| = |B|$, i.e. if A and B are identical up to a change of names.*

We define as usual an involution $\overline{(\cdot)} : \mathcal{F} \to \mathcal{F}$ (resp. $\overline{(\cdot)} : \mathcal{F}^{\mathcal{N}} \to \mathcal{F}^{\mathcal{N}}$) over propositional formulas and named formulas, expressing negation through De Morgan's duality:

$$\begin{array}{lll} \overline{(\alpha)} = \overline{\alpha} \qquad & \overline{F \vee G} = \overline{F} \wedge \overline{G} \qquad & \overline{F \wedge G} = \overline{F} \vee \overline{G}; \\ \overline{(\alpha^x)} = \overline{\alpha}^x \qquad & \overline{A \vee B} = \overline{A} \wedge \overline{B} \qquad & \overline{A \wedge B} = \overline{A} \vee \overline{B}. \end{array}$$

Definition 2 (Named sequents). Named classical propositional sequents *are expressions of the form $\vdash \Gamma, \vdash \Delta, \ldots$ where $\Gamma, \Delta, \ldots$ are* finite sets *of named formulas. Write $\Gamma \equiv \Delta$ (resp. $(\vdash \Gamma) \equiv (\vdash \Delta)$) if sets Γ, Δ (resp. sequents $\vdash \Gamma, \vdash \Delta$) are identical up to a change of names, formally if there is a* bijection $\phi : \Gamma \to \Delta$ *such that $A \equiv \phi A$ for all $A \in \Gamma$.*

For every named formula $A \in \mathcal{F}^{\mathcal{N}}$, let $\mathtt{names}(A)$ denote the set of all names occurring in A, with the obvious inductive definition. The definition extends easily to sets and named sequents by taking unions over all their members, i.e.

$$\mathtt{names}(\Gamma) = \mathtt{names}(\vdash \Gamma) = \bigcup_{A \in \Gamma} \mathtt{names}(A),$$

where Γ is any finite set of named formulas.

Definition 3 (Sharing-free formulas, sets, sequents). *Say that named formulas A, B (resp. sets Γ, Δ of formulas, sequents $\vdash \Gamma, \vdash \Delta$)* share names *if their name sets overlap (i.e. have non-empty intersection), otherwise say that they* share no names*; in the case of formulas we may also say that they are* disjoint*, while we shall not use this terminology with sets and sequents to avoid confusion with the set-theoretical meaning of the term.*

Call a named formula A sharing-free *if each name appears at most once in A; formally, by structural induction, if either A is atomic or it is of the form $B \vee C$, $B \wedge C$ where B, C are disjoint and themselves sharing-free. Call a set Γ of named formulas* sharing-free *if all formulas in Γ are sharing-free and pairwise disjoint. Call a sequent $\vdash \Gamma$* sharing-free *iff so is the set Γ.*

Definition 4 (Atom indexing notation). *Let A (resp. Γ) be a* sharing-free *named formula (resp. set of named formulas). By definition 3 there is for each name $x \in \mathtt{names}(A)$ (resp. $\mathtt{names}(\Gamma)$) a unique atom $\alpha \in \mathcal{A}$ such that α^x is a subformula of A (resp. of Γ). We write $A[x]$ (resp. $\Gamma[x]$) to denote that unique α.*

Note that negation preserves names when applied to named formulas, hence for all $A \in \mathcal{F}^{\mathcal{N}}$ we have $\mathtt{names}(A) = \mathtt{names}(\overline{A})$, and $\overline{A}[x] = \overline{A[x]}$ for all $x \in \mathtt{names}(A)$.

From now on we are going to assume that *all* named formulas and sequents be sharing-free. As a consequence, when writing down sequents through the customary comma notation:

$$\vdash \Gamma, A, B, \Delta, \ldots$$

we shall assume implicitly that every pair of comma-separated components share no names. Observe that disjoint formulas are necessarily distinct, and therefore any pair of sets of named formulas sharing no names must also be disjoint in the set-theoretical sense. In particular, in the example above, Γ, Δ are disjoint, $A \notin \Gamma$, and so on...

Remark 1. While named sequents are defined as sets, because of the presence of names this is by no means equivalent to the standard *sequents-as-sets* approach: for example, the sequent

$$\vdash \alpha^x \wedge \beta^y, \alpha^z \wedge \beta^w$$

contains two occurrences of the same anonymous formula, distinguished by their names. In other words, stripping named sequents of all names results in multisets of formulas.

We now redefine the usual `GS4` sequent system using named sequents in place of traditional ones. As announced before, we also enrich the system with *deterministic axioms* and a *superposition rule.*

Definition 5. Named `GS4` derivations *are finite trees whose nodes (also called* rule applications*) are labeled by an inference rule and by a* sharing-free *named sequent (called* conclusion *of the rule), inductively constructed in accordance with the following rules:*

– *identity rules:*

$$\frac{}{\vdash \Gamma, A, \overline{B}}\,\mathtt{ax}_{\{A,\overline{B}\}} \qquad \frac{\vdash \Gamma, A \quad \vdash \Gamma, \overline{A}}{\vdash \Gamma}\,\mathtt{cut}$$

where $A \equiv B$;

– *superposition rule:*

$$\frac{\vdash \Gamma \quad \vdash \Gamma}{\vdash \Gamma}\sqcup$$

– *logical rules:*

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}\vee \qquad \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\wedge$$

Let $\mathtt{GS4}^{\mathcal{N}}$ *denote the set of all named derivations; we use letters* $P, Q, R, \ldots$ *to range over* $\mathtt{GS4}^{\mathcal{N}}$*. We let* $\mathtt{names}(P)$ *denote the set of all names occurring in some formula in* P*.*

Remark 2. The peculiar form of axiom rules is due to the fact that the pairs of formulas they relate must share no names: because named formulas A and $\overline{A}$ share the same names, the conclusion of an hypothetical derivation tree of the form

$$\frac{}{\vdash \Gamma, A, \overline{A}}\mathtt{ax}_{\{A,\overline{A}\}}$$

would not be sharing-free, as required by definition 5.

Correctness, completeness and cut admissibility results apply as usual: by forgetting names one obtains a sound proof in the negative presentation of sequent calculus as a tree of multisets of formulas, while completeness and cut admissibility can be obtained either by adding names to anonymous proofs, or more easily by a straightforward adaptation of Schutte's completeness proof (see [31, 1]).

3 Inversion and isolation of logical rules

The foremost property of $\mathtt{GS4}^{\mathcal{N}}$ derivations is the invertibility of all logical inference rules. As is well-known, the reason why they are called *invertible* is not just that their conclusions logically entail their premises; in fact, a stronger property holds: each derivation P of $\vdash \Gamma, A$ with A non-atomic may be rewritten in such a way as to recover derivations in the context Γ of each premiss of the logical rule introducing A. We call this rewriting step *inversion*. One may then apply the logical rule again to obtain a derivation P' of $\vdash \Gamma, A$ where the last rule application introduces A: we say that P' has been obtained by *isolating* A in P. We state below the relevant facts and introduce some notation; the rewriting procedures are described in detail in appendix C.

Lemma 1. *For each derivation* $P \in \mathtt{GS4}^{\mathcal{N}}$ *with conclusion* $\vdash \Gamma, A \vee B$*, there is a derivation* $\mathtt{inv}(P, A \vee B) \in \mathtt{GS4}^{\mathcal{N}}$ *with conclusion* $\vdash \Gamma, A, B$*, cut-free if so is* P*.*

Corollary 1. *For each derivation* $P \in \mathtt{GS4}^{\mathcal{N}}$ *with conclusion* $\vdash \Gamma, A \vee B$*, there is a derivation* $\mathtt{isl}(P, A \vee B) \in \mathtt{GS4}^{\mathcal{N}}$ *with the same conclusion, cut-free if so is* P*, and whose last rule application introduces* $A \vee B$*.*

Proof. Let $\mathtt{isl}(P, A \vee B)$ be the derivation

$$\dfrac{\dfrac{\vdots\ \mathtt{inv}(P, A \vee B)}{\vdash \Gamma, A, B}}{\vdash \Gamma, A \vee B}\ \vee$$

□

Lemma 2. *For each derivation $P \in \mathtt{GS4}^{\mathcal{N}}$ with conclusion $\vdash \Gamma, A \wedge B$, there is a derivation $\mathtt{inv_l}(P, A \wedge B) \in \mathtt{GS4}^{\mathcal{N}}$ with conclusion $\vdash \Gamma, A$, cut-free if so is P.*

Lemma 3. *For each derivation $P \in \mathtt{GS4}^{\mathcal{N}}$ with conclusion $\vdash \Gamma, A \wedge B$, there is a derivation $\mathtt{inv_r}(P, A \wedge B) \in \mathtt{GS4}^{\mathcal{N}}$ with conclusion $\vdash \Gamma, B$, cut-free if so is P.*

Corollary 2. *For each derivation $P \in \mathtt{GS4}^{\mathcal{N}}$ with conclusion $\vdash \Gamma, A \wedge B$, there is a derivation $\mathtt{isl}(P, A \wedge B) \in \mathtt{GS4}^{\mathcal{N}}$ with the same conclusion, cut-free if so is P, and whose last rule application introduces $A \wedge B$.*

Proof. Let $\mathtt{isl}(P, A \wedge B)$ be the derivation

$$\dfrac{\begin{array}{c}\vdots\ \mathtt{inv_l}(P, A \wedge B) \\ \vdash \Gamma, A\end{array} \qquad \begin{array}{c}\vdots\ \mathtt{inv_r}(P, A \wedge B) \\ \vdash \Gamma, B\end{array}}{\vdash \Gamma, A \wedge B}\ \wedge$$

□

3.1 Admissibility of contraction and weakening

Lemma 4. *For each derivation $P \in \mathtt{GS4}^{\mathcal{N}}$ with conclusion $\vdash \Gamma$ and* sharing-free *set Δ of named formulas such that Γ, Δ share no names, there is a derivation $\mathtt{wk}(P, \Delta) \in \mathtt{GS4}^{\mathcal{N}}$ with conclusion $\vdash \Gamma, \Delta$, cut-free if so is P.*

The contraction rule is derivable through the use of cuts, and therefore admissible:

$$\dfrac{\begin{array}{c}\vdots\ P \\ \vdash \Gamma, A, B\end{array} \qquad \dfrac{}{\vdash \Gamma, A, \overline{B}}\ \mathtt{ax}_{\{A, \overline{B}\}}}{\vdash \Gamma, A}\ \mathtt{cut} \qquad (A \equiv B)$$

On the other hand, the weakening rule is not derivable because of the way the cut rule is formulated in GS4 (i.e. with context sharing).

4 Axiom graphs

We are now ready to formalize the idea of axiom-induced graphs. We shall rely on a standard notion of simple graph: the related notation and properties are summarized in appendix A. We start from the notion of *name graph*, i.e. a simple graph whose vertex set is a subset of the set $\mathcal{N}$ of names. We associate a name graph to each derivation; following the intuitions given in the introduction, only axioms and cuts will receive a special treatment, while all other rules shall be interpreted trivially by graph unions.

Definition 6 (Weakening and identity graphs). *For any sharing-free set Γ of named formulas, let $\mathtt{Wk}_\Gamma$ denote the graph*

$$\mathtt{Wk}_\Gamma = \langle \mathtt{names}(\Gamma), \emptyset \rangle$$

whose vertices are the names occurring in Γ and whose edge set is empty. We call $\mathtt{Wk}_\Gamma$ the edgeless *or* weakening graph *on Γ.*

For any pair $A, \overline{B}$ of disjoint and sharing-free named formulas such that $A \equiv B$, we define by induction on the height[4] *of A the graph $\mathtt{Id}_{\{A,\overline{B}\}}$:*

$$\mathtt{Id}_{\{\alpha^x,\overline{\alpha}^y\}} = \langle \{x, y\}, \{xy\} \rangle;$$

$$\mathtt{Id}_{\{A_1 \vee A_2, \overline{B}_1 \wedge \overline{B}_2\}} = \mathtt{Id}_{\{A_1 \wedge A_2, \overline{B}_1 \vee \overline{B}_2\}} = \mathtt{Id}_{\{A_1,\overline{B}_1\}} \sqcup \mathtt{Id}_{\{A_2,\overline{B}_2\}}.$$

We call $\mathtt{Id}_{\{A,\overline{B}\}}$ the identity graph *on $A, \overline{B}$. It is easy to check by induction on A that $V_{\mathtt{Id}_{\{A,\overline{B}\}}} = \mathtt{names}(\{A, \overline{B}\})$.*

In order to interpret cuts, we need to implement the informal idea of paths alternating between the two cut subderivations through the cut-formula. The implementation is independent of the logical framework and can be specified directly in terms of name graphs:

Definition 7 (Alternating paths). *Let G, H be arbitrary name graphs, $I \subseteq \mathcal{N}$ a set of names called* interface. *Furthermore, let $x_1, \ldots, x_n \in V_G \cup V_H$ be a sequence of* pairwise distinct *vertices from G, H (for some $n > 0$), and let e_i denote the unordered pair $x_i x_{i+1}$ for all $1 \leq i < n$.*

$x_1, \ldots, x_n$ is an alternating path *between G and H* through the interface *I if and only if*

(i) $x_i \in I$ for all $1 < i < n$ (those we call internal vertices *of the path);*
(ii) either $e_i \in E_G$ for all odd $1 \leq i < n$ and $e_i \in E_H$ for all even $1 \leq i < n$, or $e_i \in E_H$ for all odd $1 \leq i < n$ and $e_i \in E_G$ for all even $1 \leq i < n$.

In other words, alternating paths are built by choosing 'adjacent' edges alternately in G and H or *vice versa*, under the restriction that all vertices except the first and the last be in the interface. Observe that cycles are ignored – since vertices in the path are required to be pairwise distinct – hence when G and H are finite, there are for any given interface only finitely many alternating paths between them.

Definition 8 (Composition of name graphs). *Given name graphs G, H and a set of names $I \subseteq \mathcal{N}$, we define the* composite of G and H on interface I *as the graph*

$$G \odot_I H = \langle V, E \rangle$$

where

$$V = (V_G \cup V_H) \setminus I$$

and for all $x \neq y \in V$, $xy \in E$ if and only if there is an alternating path $z_1, \ldots, z_n$ between G and H through the interface I such that $z_1 = x$ and $z_n = y$.

[4] Which is necessarily identical to the height of B, see definition 23 from appendix B.

From now on, given any name graphs G, H and a named formula A, we may write $G \odot_A H$ as a shorthand for $G \odot_{\mathtt{names}(A)} H$.

Definition 9 (Axiom graphs). *To each derivation tree $P \in \mathtt{GS4}^{\mathcal{N}}$ we associate a named graph $[\![P]\!]$, called* axiom graph of P, *defined by structural induction on P:*

- *if P has the form*

$$\frac{}{\vdash \Gamma, A, \overline{B}}\ \mathtt{ax}_{\{A,\overline{B}\}}$$

where $A \equiv B$, then let $[\![P]\!] = \mathtt{Wk}_\Gamma \sqcup \mathtt{Id}_{\{A,\overline{B}\}}$;
- *if P has the form*

$$\frac{\begin{matrix}\vdots\ Q \\ \vdash \Gamma, A\end{matrix} \quad \begin{matrix}\vdots\ R \\ \vdash \Gamma, \overline{A}\end{matrix}}{\vdash \Gamma}\ \mathtt{cut}$$

then let $[\![P]\!] = [\![Q]\!] \odot_A [\![R]\!]$;
- *if P has the form*

$$\frac{\begin{matrix}\vdots\ Q_1 \\ \vdash \Gamma_1\end{matrix} \quad \cdots \quad \begin{matrix}\vdots\ Q_n \\ \vdash \Gamma_n\end{matrix}}{\vdash \Gamma}\ r$$

where $r \in \{\sqcup, \vee, \wedge\}$, then let $[\![P]\!] = [\![Q_1]\!] \sqcup \ldots \sqcup [\![Q_n]\!]$.

Proposition 1. *For all derivations $P \in \mathtt{GS4}^{\mathcal{N}}$ with conclusion $\vdash \Gamma$,*

$$V_{[\![P]\!]} = \mathtt{names}(\Gamma).$$

4.1 Axiom graphs do not increase under inversion and isolation

Theorem 1. *Let $P \in \mathtt{GS4}^{\mathcal{N}}$ be any named derivation tree with conclusion $\vdash \Gamma, A$, where A is any* non-atomic *named formula: then*

$$[\![\mathtt{isl}(P, A)]\!] \sqsubseteq [\![P]\!].$$

From now on, and especially in the statements and proofs of the following lemmas, we shall write expressions of the form $[\![P]\!]\restriction_{\Gamma,A}$ as a shorthand for $[\![P]\!]\restriction_{\mathtt{names}(\Gamma,A)}$ (see definition 21 for the subgraph relation and the graph restriction notation). All omitted proofs can be found in appendix E.

Lemma 5. *For all derivations $P \in \mathtt{GS4}^{\mathcal{N}}$ with conclusion $\vdash \Gamma, A \vee B$,*

$$[\![\mathtt{inv}(P, A \vee B)]\!] = [\![P]\!].$$

Lemma 6. *For all derivations $P \in \mathtt{GS4}^{\mathcal{N}}$ with conclusion $\vdash \Gamma, A \wedge B$,*

$$[\![\mathtt{inv}_\mathtt{l}(P, A \wedge B)]\!] \sqsubseteq [\![P]\!]\restriction_{\Gamma,A} \text{ and } [\![\mathtt{inv}_\mathtt{r}(P, A \wedge B)]\!] \sqsubseteq [\![P]\!]\restriction_{\Gamma,B};$$

Proof of theorem 1. Immediate consequence of lemmas 5 and 6 together with the definition of the isl-transformation (corollaries 1 and 2). The result may also be seen as a very specific instance of a more general one by Führmann and Pym [11]. □

4.2 Axiom graphs are not invariants of isolation and normalisation

The inequality from theorem 1 can be upgraded to an equality in the case of cut-free proofs:

Proposition 2. *For all cut-free derivations $P \in \mathtt{GS4}^{\mathcal{N}}$ of the sequent $\vdash \Gamma, A$ with A non-atomic,*

$$[\![\mathtt{isl}(P, A)]\!] = [\![P]\!].$$

On the other hand, the invariance result does not hold in the case of proofs with cuts, as there are derivations whose axiom graph decreases strictly under isolation of some conjunction in their conclusions. Figure 4 shows one such counterexample, where there is an alternating path (marked in blue in fig. 4a) that is lost as soon as the conjunction in the conclusion is isolated, as shown in fig. 4b.

The unstable path in fig. 4 is precisely the one that connects names occurring on the two distinct sides of a conjunction: in fact, it is possible to prove that no such edge may exist in a cut-free derivation, hence the axiom graph construction cannot be invariant under cut-elimination.

One might be tempted to modify the construction so that conjunction-crossing paths be erased as soon as possible; this, however, is not enough to guarantee invariance under isolation. Figure 5 shows a more complex example where an alternating path is lost, whose endpoints are not separated by a conjunction. The problem in this case is that the disappearing path contains edges coming from subderivations of both conjuncts of a conjunction that appears in the conclusion. As soon as the conjunction is isolated (fig. 5b), it becomes impossible to construct such a path.

5 Branch-labeled axiom graphs

Our solution is to refine the axiom graph construction by attaching labels to each edge, whose purpose is to track the branches each edge came from in the interpreted derivation. We do this by replacing the edge set with a relation associating sets of names (representing branches of derivations) to unordered pairs of vertices (representing unoriented edges). Every edge must come from some branch: if some pair of vertices has no associated name set, then we consider there to be no edge between the two vertices. We then use the additional information to discard alternating paths composed by edges belonging to incompatible branches.

5.1 Naming branches

It is a well-known fact [27, 29] that when the axiom rule is restricted to atomic conclusions, the set of axiom rule conclusions (also called *top-sequents*) of any given GS4 derivation is uniquely determined by the conclusion of the derivation. For obvious reasons, every branch is terminated by a unique axiom-rule application and can then be named by its conclusion – in fact it is sufficient to consider the set of names that occur in the leaf's conclusion. In the presence of superposition rules the naming will not be unique, but this is not a problem as identically named branches can be collapsed into one by reducing superpositions to atomic form.

In order to ensure that branch names be stable under isolation, we need to take into account the possible expansions of non-atomic axioms: branches terminated by such rule applications correspond in fact to multiple "virtual" atomic branches, still uniquely determined by the leaf's conclusion. We need then to define the unique set $\mathtt{Br}(\Gamma)$ of atomic branch names determined by a given sequent $\vdash \Gamma$. One possible approach – followed e.g. in [29] – is to rely upon `GS4` inference rules to obtain the unique atomic decomposition of the sequent; we prefer however to provide a direct characterisation of the set $\mathtt{Br}(\Gamma)$, then show that it is compatible with the inference rules.

Definition 10 (Formula and sequent branches). *We associate inductively a set of branch names to each sharing-free named formula A as follows:*

- $\mathtt{Br}(\alpha^x) = \{\{x\}\}$;
- $\mathtt{Br}(B \vee C) = \{X \cup Y \mid X \in \mathtt{Br}(B), Y \in \mathtt{Br}(C)\}$;
- $\mathtt{Br}(B \wedge C) = \mathtt{Br}(B) \cup \mathtt{Br}(C)$;

then let, for all sharing-free sets Γ of named formulas,

$$\mathtt{Br}(\Gamma) = \{X \subseteq \mathtt{names}(\Gamma) \mid \forall A \in \Gamma.\, (X \cap \mathtt{names}(A)) \in \mathtt{Br}(A)\}.$$

The construction for sets is meant to treat them as generalized disjunctions over their elements. While syntactic binary disjunction distinguishes between a left and a right subformula, elements of a set have no preferred ordering, hence the need for a slightly less straightforward definition. We provide for clarity an alternative characterisation of $\mathtt{Br}(\Gamma)$ (as usual we provide detailed proofs in appendix F):

Lemma 7. *Let Γ be any sharing-free set of named formulas. For any branch name $X \subseteq \mathcal{N}$, $X \in \mathtt{Br}(\Gamma)$ if and only if there is a family $(X_A)_{A\in\Gamma}$ of branch names such that $X = \bigcup_{A\in\Gamma} X_A$, with $X_A \in \mathtt{Br}(A)$ for all $A \in \Gamma$.*

Lemma 8. *Let Γ, Δ be sharing free sets of named formulas that share no names: then*

$$\mathtt{Br}(\Gamma \cup \Delta) = \{X \cup Y \mid X \in \mathtt{Br}(\Gamma), Y \in \mathtt{Br}(\Delta)\}.$$

Corollary 3. $\mathtt{Br}(\Gamma) = \{X \setminus \mathtt{names}(\Delta) \mid X \in \mathtt{Br}(\Gamma \cup \Delta)\}$.

It is an easy consequence of lemma 7 that $\mathtt{Br}(\{A\}) = \mathtt{Br}(A)$. Therefore, from this point on we are going to abuse systematically the usual sequent notation and write, e.g., $\mathtt{Br}(\Gamma, A, \Delta)$ for $\mathtt{Br}(\Gamma \cup \{A\} \cup \Delta)$.

Proposition 3. *Let Γ be any sharing-free set of named formulas, A, B disjoint and sharing-free named formulas that share no name with Γ:*

(i) *if all formulas in Γ are atomic, then* $\mathtt{Br}(\Gamma) = \{\mathtt{names}(\Gamma)\}$;
(ii) $\mathtt{Br}(\Gamma, A \vee B) = \mathtt{Br}(\Gamma, A, B)$;
(iii) $\mathtt{Br}(\Gamma, A \wedge B) = \mathtt{Br}(\Gamma, A) \cup \mathtt{Br}(\Gamma, B)$;
(iv) $\mathtt{Br}(\Gamma, A)$ *and* $\mathtt{Br}(\Gamma, B)$ *are disjoint.*

5.2 Branch-labeled name graphs

Definition 11. *A branch-labeled name graph is a pair* $G = \langle V_G, \prec_G \rangle$ *where* $V_G \subseteq \mathcal{N}$ *is a set of names and*

$$\prec_G \subseteq \binom{V_G}{2} \times \mathscr{P}(V_G)$$

is a binary relation between unordered pairs of vertices and arbitrary sets of vertices of G*, such that*

$$e \prec_G X \implies e \subseteq X. \tag{$\star$}$$

For G any branch-labeled name graph (hereinafter *bl-graph* for brevity), we can define a set

$$E_G = \pi_1(\prec_G) = \{e \mid \exists X.\ e \prec_G X\}$$

of unordered pairs of vertices which we shall call the *edges* of G. We define similarly the set of *branches of* G:

$$\mathtt{Br}(G) = \pi_{\mathtt{r}}(\prec_G) = \{X \mid \exists e.\ e \prec_G X\}.$$

We read the predicate $xy \prec_G X$ as x, y *are adjacent in branch* X, or *branch* X *has the edge* xy. Condition ($\star$) ensures that edges only connect vertices belonging to the branch they originated in.

We extend the subgraph relation and the union operator to bl-graphs simply by replacing the edge set with the edge-branch relation in the definition, i.e. let

$$G \sqsubseteq H \iff V_G \subseteq V_H \text{ and } \prec_G \subseteq \prec_H$$

for all pairs G, H of bl-graphs, and

$$\bigsqcup_{i \in I} G_i = \langle \bigcup_{i \in I} V_{G_i}, \bigcup_{i \in I} \prec_{G_i} \rangle$$

where I is any index set and $(G_i)_{i \in I}$ an indexed family of bl-graphs. The restriction operator must be modified slightly so as to select branches instead of edges: for G any bl-graph and $X \subseteq \mathcal{N}$, let

$$G{\upharpoonright}_X = \langle V_G \cap X, \{(e, Y) \in \prec_G \mid Y \subseteq X\} \rangle.$$

Note that while union acts upon edges in the usual way (we have $E_{G \sqcup H} = E_G \cup E_H$), restriction might remove more edges than in the case of simple graphs: for all $e \in E_G$, $e \in E_{G{\upharpoonright}_X}$ implies $e \subseteq X$, but the converse does not hold in general.

5.3 Branch-sensitive composition

We come thus to the crux of the refined approach – the composition of bl-graphs over some interface. Remember that composition is meant to interpret cuts between derivations P, Q of conclusion $\vdash \Gamma, A$ and $\vdash \Gamma, \overline{A}$ respectively, with the cut rule having conclusion $\vdash \Gamma$. We start then with atomic branch sets $\mathtt{Br}(\Gamma, A)$, $\mathtt{Br}(\Gamma, \overline{A})$, while the branch set of the final cut-free derivation will be $\mathtt{Br}(\Gamma)$. When constructing alternating paths, we are going to use edges which come in general from different branches in P, Q, but we need to ensure that they all belong to the same branch of the cut-free derivation, otherwise they might disappear under isolation.

To this end, observe that by lemma 8 all branches $X \in \mathtt{Br}(\Gamma, A)$ (resp. $\mathtt{Br}(\Gamma, \overline{A})$) are of the form $Y \cup Z$ where $Y \in \mathtt{Br}(\Gamma)$ and $Z \in \mathtt{Br}(A)$ (resp. $\mathtt{Br}(\overline{A})$). Therefore, we have $X \setminus \mathtt{names}(A) = X \setminus \mathtt{names}(\overline{A}) \in \mathtt{Br}(\Gamma)$ (corollary 3). The idea is then to check that all edges forming an alternating path share the same branch label up to names in the composition interface.

For any bl-graph G and set $I \subseteq \mathcal{N}$ of names, we define an *interface-relativized* edge-branch relation

$$\prec^I_G = \{(e, X \setminus I) \mid e \prec_G X\},$$

or equivalently

$$e \prec^I_G X \iff \exists Y.\, e \prec_G Y \text{ and } X = Y \setminus I.$$

Definition 12 (Alternating labeled paths). *Let G, H be arbitrary bl-graphs, $I, X \subseteq \mathcal{N}$ sets of names, $x_1, \ldots, x_n \in V_G \cup V_H$ a sequence of pairwise distinct vertices from G, H (with $n > 1$), and let e_i denote the unordered pair $x_i x_{i+1}$ for all $1 \leq i < n$.*

$x_1, \ldots, x_n$ is an alternating X-labeled path *between G and H through the interface I if and only if*

(i) $x_i \in I$ for all $1 < i < n$ (all internal vertices belong to the interface);
(ii) either $e_i \prec^I_G X$ for all odd $1 \leq i < n$ and $e_i \prec^I_H X$ for all even $1 \leq i < n$, or $e_i \prec^I_H X$ for all odd $1 \leq i < n$ and $e_i \prec^I_G X$ for all even $1 \leq i < n$.

We call the path complete *iff $x_1, x_n \notin I$.*

Lemma 9. *If $z_1, \ldots, z_n$ is a complete X-labeled alternating path between bl-graphs G, H through interface I, then $z_1, z_n \in X$.*

Proof. By definition 12 $n > 1$, hence there is $K \in \{G, H\}$ and $Y \in \mathtt{Br}(K)$ such that $z_1 z_2 \prec_K Y$ with $X = Y \setminus I$. By definition 11 $z_1 z_2 \subseteq Y$, and because $z_1 \notin I$ by hypothesis, we must have $z_1 \in X$. Similar reasoning shows that $z_n \in X$. $\square$

Definition 13 (Composition of bl-graphs). *Let G, H be arbitrary bl-graphs, $I \subseteq \mathcal{N}$ a set of names. We define the* composite of G and H on interface I *as the bl-graph*

$$G \odot_I H = \langle V, \prec \rangle$$

where

$$V = (V_G \cup V_H) \setminus I$$

and for all $x \neq y \in V$ and $X \subseteq V$, $xy \prec X$ if and only if there is a complete alternating X-labeled path $z_1, \ldots, z_n$ between G and H through the interface I, such that $z_1 = x$ and $z_n = y$. Lemma 9 guarantees that $xy \subseteq X$.

5.4 Interpreting derivations

Finally, we define the new inductive interpretation function for derivations. Special attention must be paid to the weakening case: it is not possible to handle it by a simple graph union, like in the original axiom graph construction, as we need to update all branch labels to take weakened formulas into account. Weakenings are then interpreted by an operator on bl-graphs. Identities also need to be tweaked to account for the way non-atomic axioms are expanded by the inversion procedures.

Definition 14 (Weakening and identities). *For any bl-graph G and sharing-free set Γ of named formulas, let*

$$\mathtt{Wk}^{\mathtt{bl}}_{\Gamma}(G) = \langle V_G \cup \mathtt{names}(\Gamma), \{(e, X \cup Y) \mid e \prec_G X,\, Y \in \mathtt{Br}(\Gamma)\}\rangle.$$

For any pair $A, \overline{B}$ of disjoint and sharing-free named formulas such that $A \equiv B$, we define by induction on the height of A the bl-graph $\mathtt{Id}^{\mathtt{bl}}_{\{A,\overline{B}\}}$:

$$\mathtt{Id}^{\mathtt{bl}}_{\{\alpha^x,\overline{\alpha}^y\}} = \langle \{x, y\}, \{(xy, \{x, y\})\}\rangle;$$
$$\mathtt{Id}^{\mathtt{bl}}_{\{A_1\vee A_2,\overline{B}_1\wedge\overline{B}_2\}} = \mathtt{Wk}^{\mathtt{bl}}_{A_2}(\mathtt{Id}^{\mathtt{bl}}_{\{A_1,\overline{B}_1\}}) \sqcup \mathtt{Wk}^{\mathtt{bl}}_{A_1}(\mathtt{Id}^{\mathtt{bl}}_{\{A_2,\overline{B}_2\}}).$$

Definition 15 (Branch-labeled axiom graphs). *To each derivation tree $P \in \mathtt{GS4}^{\mathcal{N}}$ we associate a bl-graph $(\!|P|\!)$, called* branch-labeled axiom graph of *P, defined by structural induction on P: if P has the form*

$$\frac{}{\vdash \Gamma, A, \overline{B}}\,\mathtt{ax}_{\{A,\overline{B}\}}$$

where $A \equiv B$, then let $(\!|P|\!) = \mathtt{Wk}^{\mathtt{bl}}_{\Gamma}(\mathtt{Id}^{\mathtt{bl}}_{\{A,\overline{B}\}})$; as for simple axiom graphs, cuts are interpreted by composition and all other rules by taking the bl-graph union over their subderivations.

Proposition 4. *For all derivations $P \in \mathtt{GS4}^{\mathcal{N}}$ with conclusion $\vdash \Gamma$,*

$$V_{(\!|P|\!)} = \mathtt{names}(\Gamma) \text{ and } \mathtt{Br}((\!|P|\!)) \subseteq \mathtt{Br}(\Gamma).$$

6 Main results

6.1 Behaviour under inversion and isolation

Theorem 2. *For all derivations $P \in \mathtt{GS4}^{\mathcal{N}}$ with conclusion $\vdash \Gamma, A$, where A is any* non-atomic *named formula,*

$$(\!|\mathtt{isl}(P, A)|\!) = (\!|P|\!).$$

The proof of theorem 2 rests upon the following three lemmas, whose proofs are detailed in appendix F.

Lemma 10. *For all derivations $P \in \mathtt{GS4}^{\mathcal{N}}$ with conclusion $\vdash \Gamma, A \vee B$,*

$$(\!|\mathtt{inv}(P, A \vee B)|\!) = (\!|P|\!).$$

Lemma 11. *For all derivations $P \in \mathtt{GS4}^{\mathcal{N}}$ with conclusion $\vdash \Gamma, A \wedge B$,*

$$(\!|\mathtt{inv_l}(P, A \wedge B)|\!) = (\!|P|\!)\restriction_{\Gamma,A} \text{ and } (\!|\mathtt{inv_r}(P, A \wedge B)|\!) = (\!|P|\!)\restriction_{\Gamma,B};$$

Lemma 12. *For all derivations $P \in \mathtt{GS4}^{\mathcal{N}}$ with conclusion $\vdash \Gamma, A \wedge B$,*

$$(\!|P|\!) = (\!|P|\!)\restriction_{\Gamma,A} \cup (\!|P|\!)\restriction_{\Gamma,B}.$$

Proof of theorem 2. Immediate by lemma 10 if A is a disjunction. If instead $A = B \wedge C$ is a conjunction, we have by construction[5]

$$(\!|\mathtt{isl}(P, A)|\!) = (\!|\mathtt{inv_l}(P, A \wedge B)|\!) \sqcup (\!|\mathtt{inv_r}(P, A \wedge B)|\!)$$

and then by lemmas 11 and 12

$$(\!|\mathtt{inv_l}(P, A \wedge B)|\!) \sqcup (\!|\mathtt{inv_r}(P, A \wedge B)|\!) = (\!|P|\!)\restriction_{\Gamma,B} \sqcup (\!|P|\!)\restriction_{\Gamma,C} = (\!|P|\!). \qquad \square$$

6.2 Cut-elimination theorem

Theorem 3. *For all derivations $P \in \mathtt{GS4}^{\mathcal{N}}$ with conclusion $\vdash \Gamma$, there is a* cut-free *derivation $Q \in \mathtt{GS4}^{\mathcal{N}}$ with conclusion $\vdash \Gamma$ and such that $(\!|P|\!) = (\!|Q|\!)$.*

Since we lack a cut-reduction procedure compatible with the interpretation, we have to prove theorem 3 through a *normalisation-by-evaluation* argument, where we first compute the interpretation of the derivation, then reconstruct a cut-free derivation with the same interpretation.

However, thanks to theorem 2, we can employ the isolation procedure to commute cuts up the derivation until they are reduced to atomic contexts. We are thus able to limit the need for evaluation to the very specific and much simpler case of *quasi-cut-free* derivations[6] with atomic conclusion.

The proof then looks like a standard cut-elimination argument, with a lemma for the quasi-cut-free case and a final general argument by induction on the height of the derivation. We start with a kind of semantic cut-admissibility result. All omitted proofs are provided as usual in appendix F, except that of lemma 14 to which we devote the whole of appendix G because of its complexity.

Lemma 13. *Let $P \in \mathtt{GS4}^{\mathcal{N}}$ be any derivation with conclusion $\vdash \Gamma$. All edges in $(\!|P|\!)$ link dual atom occurrences, i.e. for all $xy \in E_{(\!|P|\!)}$ we have $\Gamma[x] = \overline{\Gamma[y]}$.*

Lemma 14 (Semantic cut-admissibility). *Let $P, Q \in \mathtt{GS4}^{\mathcal{N}}$ be cut-free derivations with conclusion $\vdash \Gamma, A$ and $\vdash \Gamma, \overline{A}$ respectively, where all elements of the context Γ are atomic formulas. Then the composite bl-graph $(\!|P|\!) \odot_A (\!|Q|\!)$ has at least one edge.*

[5] See the proof of corollary 2.

[6] I.e., those formed by cutting together two cut-free derivations

Lemma 15 (Normalisation by evaluation). *Let $P, Q \in \mathtt{GS4}^{\mathcal{N}}$ be cut-free derivations with conclusion $\vdash \Gamma, A$ and $\vdash \Gamma, \overline{A}$ respectively, where all elements of the context Γ are atomic formulas. There is a cut-free derivation $R \in \mathtt{GS4}^{\mathcal{N}}$ with conclusion $\vdash \Gamma$ and such that $(\!|R|\!) = (\!|P|\!) \odot_A (\!|Q|\!)$.*

Proof. For brevity, let $G = (\!|P|\!) \odot_A (\!|Q|\!)$. Observe that the vertex set of G is finite by proposition 4, hence G has finitely many edges. Moreover, $\mathtt{Br}(G) \subseteq \mathtt{Br}(\Gamma) = \{\mathtt{names}(\Gamma)\}$ (proposition 3), i.e. all edges have the same branch label $\mathtt{names}(\Gamma)$. Finally, by lemma 13 and the assumption that Γ is atomic, there are for all edges $xy \in E_G$ named formulas $\alpha^x, \overline{\alpha}^y \in \Gamma$ where $\alpha = \Gamma[x]$ and $\overline{\alpha} = \Gamma[y]$.

Now let $|E_G|$ denote the number of edges in G: by lemma 14 we know that $|E_G| > 0$. We construct the derivation R by induction on n:

- if $|E_G| = 1$, then there is a unique edge $xy \in E_G$: let $\alpha = \Gamma[x]$, and let R be the derivation

$$\dfrac{}{\vdash \Gamma}\,\mathtt{ax}_{\{\alpha^x, \overline{\alpha}^y\}}$$

- if $|E_G| > 1$, then there is at least one edge $xy \in E_G$ and we can decompose G as $G' \sqcup G''$, where

$$G' = \langle V_G, \{(xy, \mathtt{names}(\Gamma))\}\rangle \text{ and } G'' = \langle V_G, \prec_G \setminus \{(xy, \mathtt{names}(\Gamma))\}\rangle.$$

  We have obviously $|E_{G'}|, |E_{G''}| < |E_G|$, hence there are by induction hypothesis derivations R', R'' with conclusion $\vdash \Gamma$ and such that $(\!|R'|\!) = G'$, $(\!|R''|\!) = G''$: let then R be the derivation

$$\dfrac{\begin{array}{c}\vdots\; R' \\ \vdash \Gamma\end{array} \qquad \begin{array}{c}\vdots\; R'' \\ \vdash \Gamma\end{array}}{\vdash \Gamma}\,\sqcup$$

Readers may check easily – using the facts recalled above – that $(\!|R|\!) = G$, as required. $\square$

We come finally to the general cut-elimination proof, but first we must handle an important technical detail. Because the isolation procedure might expand axioms, it does not preserve in general the height of the derivation, hence we cannot proceed by induction on that measure. We define instead an alternative measure called *virtual height*, which provides an upper bound to the height that may be attained by expanding the axioms, and therefore does not increase under isolation:

Definition 16. *For any derivation $P \in \mathtt{GS4}^{\mathcal{N}}$ with conclusion $\vdash \Gamma$,*

- *if P ends with an axiom rule application, let $\mathtt{vh}(P) = 1 + \mathtt{deg}(\Gamma)$ (definition 24);*

– *otherwise P has the form*

$$\dfrac{\begin{array}{c}\vdots\, P_1\\ \vdash \Gamma_1\end{array} \quad \cdots \quad \begin{array}{c}\vdots\, P_n\\ \vdash \Gamma_n\end{array}}{\vdash \Gamma}\, r$$

where $r \in \{\mathtt{cut}, \sqcup, \vee, \wedge\}$*: let then* $\mathtt{vh}(P) = 1 + \max_{i=1}^{n} \mathtt{vh}(P_i)$.

Lemma 16. *Let* $P \in \mathtt{GS4}^{\mathcal{N}}$ *be any derivation with conclusion* $\vdash \Gamma, A$ *where* A *is non-atomic; then* $\mathtt{vh}(\mathtt{isl}(P, A)) \leq \mathtt{vh}(P)$.

Proof of theorem 3. If P is cut-free, let $Q = P$. Otherwise, proceed by induction on the virtual height of P:

– if P has the form

$$\dfrac{\begin{array}{c}\vdots\, P_1\\ \vdash \Gamma_1\end{array} \quad \cdots \quad \begin{array}{c}\vdots\, P_n\\ \vdash \Gamma_n\end{array}}{\vdash \Gamma}\, r$$

where $r \in \{\sqcup, \vee, \wedge\}$, then apply the induction hypothesis to $P_1, \ldots, P_n$ to get cut-free derivations $Q_1, \ldots, Q_n$ with the same conclusion as $P_1, \ldots, P_n$ respectively, and let Q be the cut-free derivation

$$\dfrac{\begin{array}{c}\vdots\, Q_1\\ \vdash \Gamma_1\end{array} \quad \cdots \quad \begin{array}{c}\vdots\, Q_n\\ \vdash \Gamma_n\end{array}}{\vdash \Gamma}\, r$$

We have $\llparenthesis Q_i \rrparenthesis = \llparenthesis P_i \rrparenthesis$ for all $1 \leq i \leq n$ and therefore $\llparenthesis Q \rrparenthesis = \bigsqcup_{i=1}^{n} \llparenthesis Q_i \rrparenthesis = \bigsqcup_{i=1}^{n} \llparenthesis P_i \rrparenthesis = \llparenthesis P \rrparenthesis$, as required;

– if P has the form

$$\dfrac{\begin{array}{c}\vdots\\ \vdash \Gamma, B, A\end{array} \quad \begin{array}{c}\vdots\\ \vdash \Gamma, B, \overline{A}\end{array}}{\vdash \Gamma, B}\, \mathtt{cut}$$

where B is non-atomic, there is a derivation $P' = \mathtt{isl}(P, B)$ of the form

$$\dfrac{\begin{array}{c}\vdots\, P'_1\\ \vdash \Gamma_1\end{array} \quad \cdots \quad \begin{array}{c}\vdots\, P'_n\\ \vdash \Gamma_n\end{array}}{\vdash \Gamma, B}\, r$$

where the last rule application $r \in \{\vee, \wedge\}$ introduces the formula B, and such that $\llparenthesis P' \rrparenthesis = \llparenthesis P \rrparenthesis$ (theorem 2) and $\mathtt{vh}(P') \leq \mathtt{vh}(P)$ (lemma 16): we can then apply the induction hypothesis to the P'_i and proceed as in the previous case to get a cut-free derivation Q such that $\llparenthesis Q \rrparenthesis = \llparenthesis P' \rrparenthesis = \llparenthesis P \rrparenthesis$;

– finally, if P has the form

$$\dfrac{\dfrac{\vdots\, P_1}{\vdash \Gamma, A} \quad \dfrac{\vdots\, P_2}{\vdash \Gamma, \overline{A}}}{\vdash \Gamma}\,\texttt{cut}$$

where all elements of Γ are atomic formulas, we apply first the induction hypothesis to P_1, P_2 to get cut-free derivations Q_1, Q_2 with the same conclusions and such that $(\!|Q_i|\!) = (\!|P_i|\!)$ for $i \in \{1,2\}$. We can now apply lemma 15 to Q_1, Q_2 to get a cut-free derivation Q such that $(\!|Q|\!) = (\!|Q_1|\!) \odot_A (\!|Q_2|\!) = (\!|P_1|\!) \odot_A (\!|P_2|\!) = (\!|P|\!)$. □

7 Totality and canonical forms

The availability of a cut-elimination procedure that preserves branch-labeled axiom graphs unlocks a powerful technique for proving properties of the interpretation: we can reason about cut-free derivations and the result generalizes immediately to all derivations. An important example is the following property, which together with the one described in lemma 13 characterises the class of bl-graphs induced by derivations in $\texttt{GS4}^{\mathcal{N}}$.

Lemma 17. *For all* cut-free *derivations $P \in \texttt{GS4}^{\mathcal{N}}$ with conclusion $\vdash \Gamma$,*

$$\texttt{Br}((\!|P|\!)) = \texttt{Br}(\Gamma).$$

Corollary 4. *For all derivations $P \in \texttt{GS4}^{\mathcal{N}}$ with conclusion $\vdash \Gamma$,*

$$\texttt{Br}((\!|P|\!)) = \texttt{Br}(\Gamma).$$

Proof. Theorem 3 guarantees the existence of a cut-free derivation Q such that $(\!|Q|\!) = (\!|P|\!)$, to which we can apply lemma 17. □

Definition 17 (Totality). *Call a bl-graph G* total *w.r.t. a sharing-free named sequent $\vdash \Gamma$ if and only if*

(i) $V_G = \texttt{names}(\Gamma)$ *(G is a bl-graph on the names of Γ);*
(ii) $\texttt{Br}(G) = \texttt{Br}(\Gamma)$ *(the branches of G are those of Γ);*
(iii) for all $xy \in E_G$, $\Gamma[x] = \overline{\Gamma[y]}$ *(the edges of G link dual atoms).*

Corollary 5. *For all derivations $P \in \texttt{GS4}^{\mathcal{N}}$ with conclusion $\vdash \Gamma$, $(\!|P|\!)$ is total w.r.t. $\vdash \Gamma$.*

Proof. Immediate consequence of proposition 4, corollary 4, and lemma 13. □

Theorem 4. *Let G be any bl-graph that is total w.r.t. a sharing-free named sequent $\vdash \Gamma$: there is a derivation $P \in \texttt{GS4}^{\mathcal{N}}$ with conclusion $\vdash \Gamma$ and such that $(\!|P|\!) = G$.*

Lemma 18. *Any bl-graph G total w.r.t. $\vdash \Gamma, A \vee B$ is also total w.r.t. $\vdash \Gamma, A, B$.*

Lemma 19. *Any bl-graph G total w.r.t. $\vdash \Gamma, A \wedge B$ is decomposable as*

$$G = G\restriction_{\Gamma,A} \sqcup G\restriction_{\Gamma,B},$$

with $G\restriction_{\Gamma,A}$ and $G\restriction_{\Gamma,B}$ total w.r.t. $\vdash \Gamma, A$ and $\vdash \Gamma, B$, respectively.

Proof of theorem 4. By induction on the complexity degree of $\vdash \Gamma$ (definition 24, appendix B). Observe first that Γ cannot be empty, because then G would be the empty graph and we would have $\mathtt{Br}(G) = \emptyset \neq \{\emptyset\} = \mathtt{Br}(\Gamma)$, contradicting the totality of G w.r.t. Γ; then:

- if $\vdash \Gamma$ only contains atomic formulas, we can perform the construction described in the proof of lemma 15;
- if $\vdash \Gamma = \vdash \Delta, A \vee B$, then G is total w.r.t. $\vdash \Delta, A, B$ by lemma 18: we apply the induction hypothesis to get a derivation Q with conclusion $\vdash \Delta, A, B$ and such that $(\!|Q|\!) = G$; we conclude by applying a disjunction rule to Q:

$$P = \cfrac{\begin{array}{c}\vdots\, Q\\ \vdash \Delta, A, B\end{array}}{\vdash \Delta, A \vee B}\,\vee$$

- if $\vdash \Gamma = \vdash \Delta, A \wedge B$, then $G = G\restriction_{\Delta,A} \sqcup G\restriction_{\Delta,B}$ by lemma 19, with $G\restriction_{\Delta,A}$ and $G\restriction_{\Delta,B}$ total w.r.t. $\vdash \Delta, A$ and $\vdash \Delta, B$ respectively: we apply the induction hypothesis twice to get derivations Q, R with conclusion $\vdash \Delta, A$ and $\vdash \Delta, B$ respectively, and such that $(\!|Q|\!) = G\restriction_{\Delta,A}$, $(\!|R|\!) = G\restriction_{\Delta,B}$; we conclude by applying a conjunction rule to Q, R:

$$P = \cfrac{\begin{array}{c}\vdots\, Q\\ \vdash \Delta, A\end{array} \quad \begin{array}{c}\vdots\, R\\ \vdash \Delta, B\end{array}}{\vdash \Delta, A \wedge B}\,\wedge$$

□

Corollary 6. *Let bl-graphs G, H be total w.r.t. the sequents $\vdash \Gamma, A$ and $\vdash \Gamma, \overline{A}$, respectively: then their composite $G \odot_A H$ on A is total w.r.t. the sequent $\vdash \Gamma$.*

7.1 The proof system BLG

Upon inspection of the proof above, one sees clearly that this is a kind of sequentialization theorem. The natural question then is whether total bl-graphs can provide a canonical representation of cut-free derivations up to arbitrary permutations of logical rules.

Definition 18. *Let* `BLG` *denote the set of all pairs $\langle G, \Gamma \rangle$ such that*

(i) *G is a finite bl-graph;*
(ii) *Γ is a finite sharing-free set of named formulas;*
(iii) *G is total w.r.t. $\vdash \Gamma$.*

We know by corollary 5 and theorem 4 that there exists $\langle G, \Gamma \rangle \in$ `BLG` if and only if the sequent $\vdash \Gamma$ is a classical tautology. Is `BLG` a *proof system* for classical

$$\frac{\alpha^x, \overline{\alpha}^z \quad \alpha^x, \overline{\alpha}^w \quad \alpha^y, \overline{\alpha}^z \quad \alpha^y, \overline{\alpha}^w}{\vdash \alpha^x \wedge \alpha^y, \overline{\alpha}^z \wedge \overline{\alpha}^w}$$

$$\frac{\alpha^x, \beta^z, \overline{\alpha}^v, \gamma^w \quad \alpha^x, \overline{\gamma}^u, \overline{\alpha}^v, \gamma^w \quad \overline{\beta}^y, \beta^z, \overline{\alpha}^v, \gamma^w \quad \overline{\beta}^y, \overline{\gamma}^u, \overline{\alpha}^v, \gamma^w}{\vdash \alpha^x \wedge \overline{\beta}^y, \beta^z \wedge \overline{\gamma}^u, \overline{\alpha}^v \vee \gamma^w}$$

Fig. 3: A graphical representation of BLG proofs. Each proof is drawn as a generalised inference rule, with branch labels above the line and the conclusion below. Above each branch we draw the associated edges as black lines.

propositional logic? The question is subtle: as recalled in the introduction, the mere presence of a correctness criterion is not sufficient to provide a reasonable notion of proof. We define a notion of size of a bl-graph/sequent pair and show that totality is checkable in polynomial time: BLG is therefore a proof system in the sense of Cook and Reckhow [8].

Definition 19. *For any pair* $\mathbf{G} = \langle G, \Gamma \rangle$ *where* G *is a finite bl-graph and* Γ *a finite sharing-free set of named formulas, let*

$$\mathtt{size}(\mathbf{G}) = \mathtt{size}(\vdash \Gamma) + |V_G| + \sum_{e \prec_G X} |X|,$$

where $\mathtt{size}(\vdash \Gamma)$ *is the size of the sequent (definition 24, appendix B),* $|V_G|$ *is the number of vertices in the bl-graph* G*, and the last term is the total number of vertices in the branch labels of* G*.*

Proposition 5. *For any pair* $\mathbf{G} = \langle G, \Gamma \rangle$ *where* G *is a finite bl-graph and* Γ *a finite sharing-free set of named formulas, membership in* BLG *is decidable in polynomial time in the size of* $\mathbf{G}$*.*

7.2 Properties of the system BLG

The proof system BLG enjoys very good properties. To begin with, all logical rules of $\mathtt{GS4}^{\mathcal{N}}$ are admissible and invertible:

$$\frac{\langle G, (\Gamma, A, B) \rangle \in \mathtt{BLG}}{\langle G, (\Gamma, A \vee B) \rangle \in \mathtt{BLG}} \downarrow\vee \qquad \frac{\langle G, (\Gamma, A \vee B) \rangle \in \mathtt{BLG}}{\langle G, (\Gamma, A, B) \rangle \in \mathtt{BLG}} \uparrow\vee$$

$$\frac{\langle G, (\Gamma, A) \rangle \in \mathtt{BLG} \quad \langle H, (\Gamma, B) \rangle \in \mathtt{BLG}}{\langle G \sqcup H, (\Gamma, A \wedge B) \rangle \in \mathtt{BLG}} \downarrow\wedge$$

$$\frac{\langle G, (\Gamma, A \wedge B) \rangle \in \mathtt{BLG}}{\langle G\restriction_{\Gamma, A}, (\Gamma, A) \rangle \in \mathtt{BLG}} \uparrow\wedge_{\mathtt{l}} \qquad \frac{\langle G, (\Gamma, A \wedge B) \rangle \in \mathtt{BLG}}{\langle G\restriction_{\Gamma, B}, (\Gamma, B) \rangle \in \mathtt{BLG}} \uparrow\wedge_{\mathtt{r}}$$

The cut-rule is also admissible by corollary 6:

$$\frac{\langle G, (\Gamma, A)\rangle \in \mathtt{BLG} \quad \langle H, (\Gamma, \overline{A})\rangle \in \mathtt{BLG}}{\langle G \odot_A H, \Gamma\rangle \in \mathtt{BLG}}\,\mathtt{cut}$$

Finally, axiom and superposition rules are obviously admissible through their interpretation:

$$\frac{A \equiv B}{\langle \mathtt{Wk}^{\mathtt{bl}}_{\Gamma}(\mathtt{Id}^{\mathtt{bl}}_{\{A,\overline{B}\}}), (\Gamma, A, \overline{B})\rangle \in \mathtt{BLG}}\,\mathtt{ax} \qquad \frac{\langle G, \Gamma\rangle \in \mathtt{BLG} \quad \langle H, \Gamma\rangle \in \mathtt{BLG}}{\langle G \sqcup H, \Gamma\rangle \in \mathtt{BLG}}\,\sqcup$$

Isolation is as expected an identity on BLG proofs. The inversion procedure amounts to a simple replacement of the conclusion for disjunctions; in the case of conjunctions some time must be spent computing the graph restriction, but it will be often significantly less than the time spent to perform inversion on a sequent calculus derivation, especially when all axioms in the derivation are atomic. These facts – together with the immediate notation and the availability of equational reasoning – make BLG a very efficient tool for reasoning about proof content and transformations.

The price to be paid lies in the size of proof objects, which is often exponential in the complexity of the conclusion even when much smaller derivations would be available in the context-splitting formulation of sequent calculus or even in GS4 with non-atomic axioms. This is a well-known problem that BLG shares with GS4 when restricted to atomic axioms. It is an open question whether some proof-compression method could be devised to reduce the size BLG proofs without loosing the good properties of the system.

8 On the absence of a cut-reduction procedure

By cut-reduction procedure we mean a set of *syntactical* rewriting steps capable of reducing the complexity of cut-rule applications in a derivation until they become atomic, while at the same time preserving correctness and the conclusion of the derivation. One such step is implemented in the proof of theorem 3, where we apply the isolation procedure to permute cuts towards the top of the derivation, thus reducing the complexity of their context.

At present, however, we don't know of any rewriting step capable of reducing the complexity of cut-formulas in $\mathtt{GS4}^{\mathcal{N}}$ derivations while preserving their associated bl-graph. There is a known pair of natural cut-reduction steps, considered by Pulcini in [30]:

$$\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma, A, B}}{\vdash \Gamma, A \vee B}\vee \qquad \dfrac{\overset{\vdots\, Q}{\vdash \Gamma, \overline{A}} \quad \overset{\vdots\, R}{\vdash \Gamma, \overline{B}}}{\vdash \Gamma, \overline{A} \wedge \overline{B}}\wedge}{\vdash \Gamma}\texttt{cut} \quad\longrightarrow\quad \dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma, A, B} \quad \overset{\vdots\, \texttt{wk}(Q, B)}{\vdash \Gamma, \overline{A}, B}}{\vdash \Gamma, B}\texttt{cut} \qquad \overset{\vdots\, R}{\vdash \Gamma, \overline{B}}}{\vdash \Gamma}\texttt{cut}$$

$$\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma, A, B}}{\vdash \Gamma, A \vee B}\vee \qquad \dfrac{\overset{\vdots\, Q}{\vdash \Gamma, \overline{A}} \quad \overset{\vdots\, R}{\vdash \Gamma, \overline{B}}}{\vdash \Gamma, \overline{A} \wedge \overline{B}}\wedge}{\vdash \Gamma}\texttt{cut} \quad\longrightarrow\quad \dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma, A, B} \quad \overset{\vdots\, \texttt{wk}(R, A)}{\vdash \Gamma, A, \overline{B}}}{\vdash \Gamma, A}\texttt{cut} \qquad \overset{\vdots\, Q}{\vdash \Gamma, \overline{A}}}{\vdash \Gamma}\texttt{cut}$$

These are straightforward adaptations of the usual key steps for context-splitting systems to the context-sharing setting of the system $\texttt{GS4}^{\mathcal{N}}$. The resulting cut reduction procedure is non-deterministic at the syntactical level, but both rewriting steps preserve the axiom graph obtained through the unrefined construction (definition 9).

Unfortunately, the two steps are incompatible with the refined interpretation (definition 15). To see why, one must remember that the branch-sensitive notion of composition is designed to omit all those paths that connect the two sides of conjunctions occurring *outside the interface*. The two rewriting steps above, in reducing one cut to *two* cuts of lower complexity, select one subformula of the disjunction to bring temporarily outside the interface. This might result in the loss of some paths when the selected formula is a conjunction:

$$\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma, A, B \wedge C}}{\vdash \Gamma, A \vee (B \wedge C)}\vee \qquad \dfrac{\overset{\vdots\, Q}{\vdash \Gamma, \overline{A}} \quad \overset{\vdots\, R}{\vdash \Gamma, \overline{B} \vee \overline{C}}}{\vdash \Gamma, \overline{A} \wedge (\overline{B} \vee \overline{C})}\wedge}{\vdash \Gamma}\texttt{cut} \quad\longrightarrow\quad \dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma, A, B \wedge C} \quad \overset{\vdots\, \texttt{wk}(Q, B \wedge C)}{\vdash \Gamma, \overline{A}, B \wedge C}}{\vdash \Gamma, B \wedge C}\texttt{cut} \qquad \overset{\vdots\, R}{\vdash \Gamma, \overline{B} \vee \overline{C}}}{\vdash \Gamma}\texttt{cut}$$

Here the left side of the rewriting rule is associated to the bl-graph

$$(\!|P|\!) \odot_{(A \vee (B \wedge C))} ((\!|Q|\!) \sqcup (\!|R|\!)),$$

while the right side is interpreted as

$$((\!|P|\!) \odot_A (\!|Q|\!)) \odot_{B \wedge C} (\!|R|\!);$$

the conjunction $B \wedge C$ is contained in the interface in the former expression, where the interpretation is computed by a single composition step, but lies outside the interface in the first half of the latter expression, where we compute first the intermediate step $(\!|P|\!) \odot_A (\!|Q|\!)$. To show that this is an actual problem, not just a hypothetical possibility, we provide a complete (but slightly involved) counter-example in fig. 6.

It remains unclear whether there is some syntactical counterpart to the semantical cut-elimination procedure described in section 6.2. It is possible that the problem could be solved simply by superposing the two reducts, thus making the rewriting step deterministic:

$$
\dfrac{
\dfrac{\dfrac{\begin{array}{c}\vdots\; P\\ \vdash \Gamma, A, B\end{array} \quad \begin{array}{c}\vdots\; \mathtt{wk}(Q,B)\\ \vdash \Gamma, \overline{A}, B\end{array}}{\vdash \Gamma, B}\,\mathtt{cut} \quad \begin{array}{c}\vdots\; R\\ \vdash \Gamma, \overline{B}\end{array}}{\vdash \Gamma}\,\mathtt{cut}
\qquad
\dfrac{\dfrac{\begin{array}{c}\vdots\; P\\ \vdash \Gamma, A, B\end{array} \quad \begin{array}{c}\vdots\; \mathtt{wk}(R,A)\\ \vdash \Gamma, A, \overline{B}\end{array}}{\vdash \Gamma, A}\,\mathtt{cut} \quad \begin{array}{c}\vdots\; Q\\ \vdash \Gamma, \overline{A}\end{array}}{\vdash \Gamma}\,\mathtt{cut}
}{\vdash \Gamma}\,\sqcup
$$

We have not been able to find a counter-example so far, but we have yet to attempt a proof. We would have to show somehow that whenever some path is erased on one side, then it must be preserved on the other side, and *vice versa*.

A different possibility would be to further refine the interpretation so as to make it invariant under the two traditional cut-reduction steps. The key observation in this regard is that all counter-examples known to us – including indeed the one in fig. 6 – seem to rely critically on the possibility of constructing alternating paths using edges from both branches of a superposition rule. This happens because superposition is not interpreted as a non-deterministic sum of separate proofs, but as some kind of parallel composition that "blends" two proofs together, thus obtaining a new and generally distinct one. We conjecture that by treating superpositions as proper non-deterministic sums we should be able to recover invariance under the traditional logical reduction steps. The picture however is complicated by the fact that superpositions might be introduced when reducing weakening-weakening cuts: it is not clear then how the new composition operator should look like.

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A Graphs

Definition 20 (Simple graphs). *A* graph *(also known as* simple graph*) is defined by a pair $G = \langle V_G, E_G\rangle$ where V_G is a set of* vertices *and E_G is a set of unordered pairs of vertices, i.e.* two-element subsets *of V_G, called the* edges of G. *For any pair of distinct vertices $u \neq v \in V_G$, let uv denote the set $\{u, v\}$ and say that u and v are* adjacent *in G iff $uv \in E_G$.*

Definition 21 (Subgraphs). *Given graphs H, G, say that H* is a subgraph of *G and write $H \sqsubseteq G$ iff $V_H \subseteq V_G$ and $E_H \subseteq E_G$. Write $H \sqsubset G$ iff $H \sqsubseteq G$ and $H \neq G$.*

Given a graph G, any set S of vertices induces a subgraph $H \sqsubseteq G$ by letting $V_H = V_H \cap S$ and restricting the edge set:

$$E_H = \{e \in E_G \mid e \subseteq S\}.$$

We say that H is an induced subgraph *of G and denote it by $G\restriction_S$.*

Graphs can be combined by an operation related to set-theoretic union, which constructs the least upper bound of a family of graphs w.r.t. the subgraph order $\sqsubseteq$:

Definition 22 (Graph union). *Let I be a set, $(G_i)_{i\in I}$ a family of graphs indexed by I. We define the* union *or* superposition *of the family as the graph*

$$\bigsqcup_{i\in I} G_i = \langle \bigcup_{i\in I} V_{G_i}, \bigcup_{i\in I} E_{G_i}\rangle.$$

As a special case, given $n \geq 0$ and graphs $G_1, \ldots, G_n$, we define the finite union

$$G_1 \sqcup \ldots \sqcup G_n = \bigsqcup_{i=1}^{n} G_i.$$

Proposition 6. *Let I be a set, $(G_i)_{i\in I}$ a family of graphs indexed by I; the union graph $\bigsqcup_{i\in I} G_i$ is the least upper bound of the family w.r.t. the subgraph order $\sqsubseteq$.*

Proposition 7. *Let I be any set, $(G_i)_{i\in I}$ a family of graphs indexed by I, S an arbitrary set of vertices: then*

$$\left(\bigsqcup_{i\in I} G_i\right)\restriction_S = \bigsqcup_{i\in I}(G_i\restriction_S)$$

B Complexity measures on formulas, sequents and derivations

We define in the following paragraphs some useful measures of the complexity of formulas, sequents and derivations, and discuss the relationships between them.

Definition 23 (Complexity of formulas). *We define by structural induction four different measures on named formulas A: the* height $\mathtt{h}(A)$*, the* atom count $\#\mathtt{at}(A)$*, the* degree $\mathtt{deg}(A)$ *and the* size $\mathtt{size}(A)$*:*

- *if A is atomic, let*

$$\mathtt{h}(A) = \mathtt{deg}(A) = 0,$$
$$\#\mathtt{at}(A) = \mathtt{size}(A) = 1;$$

- *if $A = B \vee C$ or $A = B \wedge C$, let*

$$\begin{aligned} \mathtt{h}(A) &= 1 + \max\{\mathtt{h}(B), \mathtt{h}(C)\},\\ \#\mathtt{at}(A) &= \#\mathtt{at}(B) + \#\mathtt{at}(C),\\ \mathtt{deg}(A) &= 1 + \mathtt{deg}(B) + \mathtt{deg}(C),\\ \mathtt{size}(A) &= 1 + \mathtt{size}(B) + \mathtt{size}(C). \end{aligned}$$

The height tracks the length of the longest branch in the syntax tree of the formula, the atom count is self-explanatory, the degree is the number of logical operators occurring in the formula, and the size is the number of symbols (atoms and operators).

Proposition 8. *For all named formulas A:*

(i) $\#\mathtt{at}(A) = 1 + \mathtt{deg}(A)$*;*
(ii) $\mathtt{size}(A) = \#\mathtt{at}(A) + \mathtt{deg}(A) = 1 + 2\cdot\mathtt{deg}(A)$*;*
(iii) $\mathtt{h}(A) \le \mathtt{deg}(A) < \#\mathtt{at}(A) \le \mathtt{size}(A)$.

Proof. The inequalities $\deg(A) < \#\mathtt{at}(A)$ and $\#\mathtt{at}(A) \le \mathtt{size}(A)$ are direct consequences of facts (i) and (ii). We prove facts (i) and (ii) and the first inequality simultaneously by structural induction on A:

- if A is atomic, then

$$0 = \mathtt{h}(A) \le \mathtt{deg}(A) = 0,$$
$$\#\mathtt{at}(A) = 1 = 1 + 0 = 1 + \mathtt{deg}(A), \text{ and}$$
$$\mathtt{size}(A) = 1 = 1 + 0 = \#\mathtt{at}(A) + \mathtt{deg}(A);$$

– if $A = B \vee C$ or $A = B \wedge C$, then

$$\begin{aligned}
\#\mathtt{at}(A) &= \#\mathtt{at}(B) + \#\mathtt{at}(C) && \text{(def. 23)}\\
&= \mathtt{deg}(B) + 1 + \mathtt{deg}(C) + 1 && \text{(ind. hyp.)}\\
&= 1 + \mathtt{deg}(B) + \mathtt{deg}(C) + 1 &&\\
&= \mathtt{deg}(A) + 1 && \text{(def. 23)}
\end{aligned}$$

$$\begin{aligned}
\mathtt{size}(A) &= 1 + \mathtt{size}(B) + \mathtt{size}(C) && \text{(def. 23)}\\
&= 1 + \#\mathtt{at}(B) + \mathtt{deg}(B) + \#\mathtt{at}(C) + \mathtt{deg}(C) && \text{(ind. hyp.)}\\
&= \#\mathtt{at}(B) + \#\mathtt{at}(C) + 1 + \mathtt{deg}(B) + \mathtt{deg}(C) &&\\
&= \#\mathtt{at}(A) + \mathtt{deg}(A) && \text{(def. 23)}
\end{aligned}$$

$$\begin{aligned}
\mathtt{h}(A) &= 1 + \max\{\mathtt{h}(B), \mathtt{h}(C)\} && \text{(def. 23)}\\
&\leq 1 + \mathtt{h}(B) + \mathtt{h}(C) &&\\
&\leq 1 + \mathtt{deg}(B) + \mathtt{deg}(C) && \text{(ind. hyp.)}\\
&= \mathtt{deg}(A) && \text{(def. 23)}
\end{aligned}$$

□

Definition 24 (Complexity of sequents). *We define for sequents the same measures as for formulas, by taking sums over all formulas contained in the sequent: for all named sequents* $\vdash \Gamma$ *let*

$$\mathtt{h}(\vdash \Gamma) = \sum_{A \in \Gamma} \mathtt{h}(A), \qquad \#\mathtt{at}(\vdash \Gamma) = \sum_{A \in \Gamma} \#\mathtt{at}(A),$$

$$\mathtt{deg}(\vdash \Gamma) = \sum_{A \in \Gamma} \mathtt{deg}(A), \qquad \mathtt{size}(\vdash \Gamma) = \sum_{A \in \Gamma} \mathtt{size}(A).$$

Since Γ *is required to be finite, all measures are well-defined.*

Proposition 9. *For all named sequents* $\vdash \Gamma$*:*

(i) $\#\mathtt{at}(\vdash \Gamma) = |\Gamma| + \mathtt{deg}(\vdash \Gamma)$*;*
(ii) $\mathtt{size}(\vdash \Gamma) = \#\mathtt{at}(\vdash \Gamma) + \mathtt{deg}(\vdash \Gamma) = |\Gamma| + 2 \cdot \mathtt{deg}(A)$*;*
(iii) $\mathtt{h}(\vdash \Gamma) \leq \mathtt{deg}(\vdash \Gamma) < \#\mathtt{at}(\vdash \Gamma) \leq \mathtt{size}(\vdash \Gamma)$*.*

Proof. For fact (i), observe that

$$\begin{aligned}
\#\mathtt{at}(\vdash \Gamma) &= \sum_{A \in \Gamma} \#\mathtt{at}(A) && \text{(def. 24)}\\
&= \sum_{A \in \Gamma} 1 + \mathtt{deg}(A) && \text{(lemma 8 (i))}\\
&= \left(\sum_{A \in \Gamma} 1\right) + \left(\sum_{A \in \Gamma} \mathtt{deg}(A)\right) && \text{(associativity)}\\
&= \#\Gamma + \mathtt{deg}(\vdash \Gamma) && \text{(def. 24).}
\end{aligned}$$

Fact (ii) follows from definition 24 and the associativity of sums; fact (iii) follows from proposition 8, point (iii) and the fact that addition is strictly monotonic over the natural numbers. □

Definition 25 (Complexity of derivations). *We define by structural induction two complexity measures on named derivation trees $P \in \mathtt{GS4}^{\mathcal{N}}$: the* height $\mathtt{h}(P)$ *and the* size $\mathtt{size}(P)$*:*

- *if P has the form*

$$\frac{}{\vdash \Gamma, A, \overline{B}}\,\mathtt{ax}_{\{A,\overline{B}\}}$$

where $A \equiv B$, then let

$$\mathtt{h}(P) = 0,\ \ \mathtt{size}(P) = 1;$$

- *otherwise P has the form*

$$\frac{\begin{array}{c}\vdots\, Q_1\\ \vdash \Gamma_1\end{array} \quad \cdots \quad \begin{array}{c}\vdots\, Q_n\\ \vdash \Gamma_n\end{array}}{\vdash \Gamma}\,r$$

where $r \in \{\mathtt{cut}, \sqcup, \vee, \wedge\}$ and $n > 0$: then let

$$\mathtt{h}(P) = 1 + \max_{1 \le i \le n} \mathtt{h}(Q_i),$$

$$\mathtt{size}(P) = 1 + \sum_{1 \le i \le n} \mathtt{size}(Q_i).$$

C Proofs of the inversion lemmas

We collect in the present section all proofs of the inversion lemmas from section 3 plus the proofs of admissibility for contraction and weakening.

Proof of lemma 1. By structural induction on P:

- if P has the form

$$\frac{}{\vdash \Delta, \overline{C} \wedge \overline{D}, A \vee B}\,\mathtt{ax}_{\{\overline{C}\wedge\overline{D}, A\vee B\}}$$

where $A \equiv C$, $B \equiv D$ and $\Gamma = \Delta \cup \{\overline{C} \wedge \overline{D}\}$, then let $\mathtt{inv}(P, A \vee B)$ be

$$\frac{\dfrac{}{\vdash \Delta, \overline{C}, A, B}\,\mathtt{ax}_{\{\overline{C},A\}} \qquad \dfrac{}{\vdash \Delta, \overline{D}, A, B}\,\mathtt{ax}_{\{\overline{D},B\}}}{\vdash \Delta, \overline{C} \wedge \overline{D}, A, B}\,\wedge$$

– if P has the form

$$\dfrac{}{\vdash \Delta, C, \overline{D}, A \vee B}\,\mathtt{ax}_{\{C,\overline{D}\}}$$

where $C \equiv D$ and $\Gamma = \Delta \cup \{C, \overline{D}\}$, then let $\mathtt{inv}(P, A \vee B)$ be

$$\dfrac{}{\vdash \Delta, C, \overline{D}, A, B}\,\mathtt{ax}_{\{C,\overline{D}\}}$$

– if P has the form

$$\dfrac{\overset{\vdots\, Q}{\vdash \Gamma, A, B}}{\vdash \Gamma, A \vee B}\,\vee$$

then let $\mathtt{inv}(P, A \vee B) = Q$;

– if P has the form

$$\dfrac{\overset{\vdots\, Q_1}{\vdash \Gamma_1, A \vee B} \quad \cdots \quad \overset{\vdots\, Q_n}{\vdash \Gamma_n, A \vee B}}{\vdash \Gamma, A \vee B}\,r$$

where $r \in \{\mathtt{cut}, \sqcup, \vee, \wedge\}$ and $n > 0$, then apply $\mathtt{inv}(-, A \vee B)$ recursively to each Q_i and let $\mathtt{inv}(P, A \vee B)$ be

$$\dfrac{\overset{\vdots\, \mathtt{inv}(Q_1, A \vee B)}{\vdash \Gamma_1, A, B} \quad \cdots \quad \overset{\vdots\, \mathtt{inv}(Q_n, A \vee B)}{\vdash \Gamma_n, A, B}}{\vdash \Gamma, A, B}\,r$$

□

Proof of lemma 2. By structural induction on P:

– if P has the form

$$\dfrac{}{\vdash \Delta, \overline{C} \vee \overline{D}, A \wedge B}\,\mathtt{ax}_{\{\overline{C} \vee \overline{D}, A \wedge B\}}$$

where $A \equiv C$, $B \equiv D$ and $\Gamma = \Delta \cup \{\overline{C} \vee \overline{D}\}$, then let $\mathtt{inv}_1(P, A \wedge B)$ be

$$\dfrac{\dfrac{}{\vdash \Delta, \overline{C}, \overline{D}, A}\,\mathtt{ax}_{\{\overline{C},A\}}}{\vdash \Delta, \overline{C} \vee \overline{D}, A}\,\vee$$

– if P has the form

$$\dfrac{}{\vdash \Delta, C, \overline{D}, A \wedge B}\,\mathtt{ax}_{\{C,\overline{D}\}}$$

where $C \equiv D$ and $\Gamma = \Delta \cup \{C, \overline{D}\}$, then let $\mathtt{inv}_1(P, A \wedge B)$ be

$$\dfrac{}{\vdash \Delta, C, \overline{D}, A}\,\mathtt{ax}_{\{C,\overline{D}\}}$$

– if P has the form

$$\dfrac{\overset{\vdots\, Q}{\vdash \Gamma, A} \quad \overset{\vdots\, R}{\vdash \Gamma, B}}{\vdash \Gamma, A \wedge B}\wedge$$

then let $\mathtt{inv_l}(P, A \wedge B) = Q$;

– if P has the form

$$\dfrac{\overset{\vdots\, Q_1}{\vdash \Gamma_1, A \wedge B} \quad \cdots \quad \overset{\vdots\, Q_n}{\vdash \Gamma_n, A \wedge B}}{\vdash \Gamma, A \wedge B}\, r$$

where $r \in \{\mathtt{cut}, \sqcup, \vee, \wedge\}$ and $n > 0$, then apply $\mathtt{inv_l}(-, A \wedge B)$ recursively to each Q_i and let $\mathtt{inv_l}(P, A \wedge B)$ be

$$\dfrac{\overset{\vdots\, \mathtt{inv_l}(Q_1, A\wedge B)}{\vdash \Gamma_1, A} \quad \cdots \quad \overset{\vdots\, \mathtt{inv_l}(Q_n, A\wedge B)}{\vdash \Gamma_n, A}}{\vdash \Gamma, A}\, r$$

□

Proof of lemma 3. By structural induction on P:

– if P has the form

$$\dfrac{}{\vdash \Delta, \overline{C} \vee \overline{D}, A \wedge B}\, \mathtt{ax}_{\{\overline{C}\vee\overline{D}, A\wedge B\}}$$

where $A \equiv C$, $B \equiv D$ and $\Gamma = \Delta \cup \{\overline{C} \vee \overline{D}\}$, then let $\mathtt{inv_r}(P, A \wedge B)$ be

$$\dfrac{\dfrac{}{\vdash \Delta, \overline{C}, \overline{D}, B}\, \mathtt{ax}_{\{\overline{D}, B\}}}{\vdash \Delta, \overline{C} \vee \overline{D}, B}\vee$$

– if P has the form

$$\dfrac{}{\vdash \Delta, C, \overline{D}, A \wedge B}\, \mathtt{ax}_{\{C, \overline{D}\}}$$

where $C \equiv D$ and $\Gamma = \Delta \cup \{C, \overline{D}\}$, then let $\mathtt{inv_r}(P, A \wedge B)$ be

$$\dfrac{}{\vdash \Delta, C, \overline{D}, B}\, \mathtt{ax}_{\{C, \overline{D}\}}$$

– if P has the form

$$\dfrac{\overset{\vdots\, Q}{\vdash \Gamma, A} \quad \overset{\vdots\, R}{\vdash \Gamma, B}}{\vdash \Gamma, A \wedge B}\wedge$$

then let $\mathtt{inv_r}(P, A \wedge B) = R$;

– if P has the form

$$\dfrac{\begin{array}{c}\vdots\, Q_1 \\ \vdash \Gamma_1, A \wedge B\end{array} \quad \cdots \quad \begin{array}{c}\vdots\, Q_n \\ \vdash \Gamma_n, A \wedge B\end{array}}{\vdash \Gamma, A \wedge B}\, r$$

where $r \in \{\mathtt{cut}, \sqcup, \vee, \wedge\}$ and $n > 0$, then apply $\mathtt{inv_r}(-, A \wedge B)$ recursively to each Q_i and let $\mathtt{inv_r}(P, A \wedge B)$ be

$$\dfrac{\begin{array}{c}\vdots\, \mathtt{inv_r}(Q_1, A \wedge B) \\ \vdash \Gamma_1, B\end{array} \quad \cdots \quad \begin{array}{c}\vdots\, \mathtt{inv_r}(Q_n, A \wedge B) \\ \vdash \Gamma_n, B\end{array}}{\vdash \Gamma, B}\, r$$

□

Proof of lemma 4. Let us fix once and for all a complete enumeration $x_1, x_2, \ldots$ *without repetitions* of the countable set $\mathcal{N}$ of names. The `wk`-transformation will be deterministic up to our choice of enumeration. We are also gonna need the theory of renamings from appendix D. We proceed by induction on the height of P:

– if P has the form

$$\dfrac{}{\vdash \Gamma', C, \overline{D}}\, \mathtt{ax}_{\{C, \overline{D}\}}$$

where $C \equiv D$ and $\Gamma = \Gamma' \cup \{C, \overline{D}\}$, then let $\mathtt{wk}(P, \Delta)$ be

$$\dfrac{}{\vdash \Gamma', C, \overline{D}, \Delta}\, \mathtt{ax}_{\{C, \overline{D}\}}$$

– if P has the form

$$\dfrac{\begin{array}{c}\vdots\, Q_1 \\ \vdash \Gamma_1\end{array} \quad \cdots \quad \begin{array}{c}\vdots\, Q_n \\ \vdash \Gamma_n\end{array}}{\vdash \Gamma}\, r$$

where $r \in \{\sqcup, \vee, \wedge\}$ and $n > 0$, then one may check easily (by inspection of the three rules listed before) that $\mathtt{names}(\Gamma_i) \subseteq \mathtt{names}(\Gamma)$ for all $1 \leq i \leq n$: apply $\mathtt{wk}(-, \Delta)$ recursively to each Q_i and let $\mathtt{wk}(P, \Delta)$ be

$$\dfrac{\begin{array}{c}\vdots\, \mathtt{wk}(Q_1, \Delta) \\ \vdash \Gamma_1, \Delta\end{array} \quad \cdots \quad \begin{array}{c}\vdots\, \mathtt{wk}(Q_n, \Delta) \\ \vdash \Gamma_n, \Delta\end{array}}{\vdash \Gamma, \Delta}\, r$$

– if P has the form

$$\dfrac{\begin{array}{c}\vdots\, Q \\ \vdash \Gamma, A\end{array} \quad \begin{array}{c}\vdots\, R \\ \vdash \Gamma, \overline{A}\end{array}}{\vdash \Gamma}\, \mathtt{cut}$$

we must take into account the possibility that $A, \overline{A}$ and Δ share some names, as the hypothesis only guarantees that Δ share no names with Γ.

Let then k be the smallest natural number such that, for all $1 \leq i \geq k$, $x_i \notin \texttt{names}(P) \cup \texttt{names}(\Delta)$: such a k must exist because only finitely many names may occur either in P or Δ, and the enumeration has no repetitions. We construct a renaming $\phi : \texttt{names}(P) \to \mathcal{N}$ of P by letting

$$\phi(x_i) = \begin{cases} x_{i+k} & \text{if } x_i \in \texttt{names}(\Delta), \\ x_i & \text{otherwise,} \end{cases}$$

for all $i \in \mathbb{N}$ such that $x_i \in \texttt{names}(P)$. Because no name in Γ occurs also in Δ, ϕ is guaranteed to act as the identity on all formulas in Γ, i.e. ϕP has the form

$$\dfrac{\dfrac{\vdots\ \phi Q}{\vdash \Gamma, \phi A} \qquad \dfrac{\vdots\ \phi R}{\vdash \Gamma, \overline{(\phi A)}}}{\vdash \Gamma}\ \texttt{cut}$$

where $\phi A, \overline{(\phi A)}$ share no location with Δ by construction of ϕ, and $\phi Q, \phi R$ have by proposition 11 the same height as Q, R respectively: let then $\texttt{wk}(P, \Delta)$ be

$$\dfrac{\dfrac{\vdots\ \texttt{wk}(\phi Q, \Delta)}{\vdash \Gamma, \phi A, \Delta} \qquad \dfrac{\vdots\ \texttt{wk}(\phi R, \Delta)}{\vdash \Gamma, \overline{(\phi A)}, \Delta}}{\vdash \Gamma, \Delta}\ \texttt{cut}$$

□

D Renamings

Definition 26. *A* renaming *of a named formula A (resp. set Γ of named formulas) is an* injective map *from* $\texttt{names}(A)$ *(resp.* $\texttt{names}(\Gamma)$*) to $\mathcal{N}$. For any named formula A and renaming ϕ of A, let*

$$\phi A = \begin{cases} \alpha^{\phi x} & \text{if } A = \alpha^x \text{ for some } \alpha \in \mathcal{A}, x \in \mathcal{N}, \\ \phi B \vee \phi C & \text{if } A = B \vee C, \\ \phi B \wedge \phi C & \text{if } A = B \wedge C. \end{cases}$$

The application of a renaming ϕ to a set Γ is defined by taking its image under ϕ, observing that, for each $A \in \Gamma$, the restriction of ϕ to $\texttt{names}(A)$ *is necessarily a renaming of A.*

Proposition 10. *For any named formula A and renaming ϕ of A, ϕA is also a named formula with $A \equiv \phi A$,* $\texttt{names}(\phi A) = \phi(\texttt{names}(A))$*, and sharing-free iff so is A.*

Proof. The equivalence result and that about name sets are more or less immediate, by structural induction on A. As regards sharing-freedom, we reason again by structural induction on A:

– if A is atomic then so is ϕA and they are both sharing-free by definition;

- if $A = B \diamond C$ (where $\diamond \in \{\vee, \wedge\}$), then $\phi A = \phi B \diamond \phi C$. By induction hypothesis B, C are sharing-free if and only if so are $\phi B, \phi C$, respectively. Moreover, because ϕ is injective, $\mathtt{names}(\phi B) = \phi(\mathtt{names}(B))$ and the same holds for C *mutatis mutandis*, B, C are disjoint if and only if so are $\phi B, \phi C$. Then A is sharing-free, by definition, iff B, C are disjoint and both sharing-free, iff $\phi B, \phi C$ are disjoint and both sharing-free, iff again by definition ϕA is sharing-free. $\square$

Lemma 20. *Let A be any named formula:*

(i) ϕ is a renaming of A if and only if it is a renaming of $\overline{A}$;
(ii) for any renaming ϕ of A, $\phi(\overline{A}) = \overline{(\phi A)}$.

Proof. Left to the reader. $\square$

Lemma 21. *Let Γ be a set of named formulas, ϕ a renaming of Γ:*

(i) for all $A, B \in \Gamma$, $A = B$ if and only if $\phi A = \phi B$;
(ii) for all $A, B \in \Gamma$, A, B share names if and only if so do $\phi A, \phi B$;
(iii) for all $\Delta, \Delta' \subseteq \Gamma$, Δ, Δ' share names if and only if so do $\phi\Delta, \phi\Delta'$.

Proof. Point (ii) is a special case of point (iii), which is in turn an immediate consequence of the injectivity of ϕ plus the obvious facts that $\mathtt{names}(\phi\Delta) = \phi(\mathtt{names}(\Delta))$ and $\mathtt{names}(\phi\Delta') = \phi(\mathtt{names}(\Delta'))$. For point (i) we proceed by structural induction on A:

- A is atomic: then $A = B$ if and only if $A = \alpha^x = B$ (resp. $A = \overline{\alpha}^x = B$) for some $\alpha \in \mathcal{A}$ and $x \in \mathcal{N}$, if and only if $\phi A = \alpha^{\phi x} = \phi B$ (resp. $\phi A = \overline{\alpha}^{\phi x} = \phi B$);
- $A = C \diamond D$ where $\diamond \in \{\vee, \wedge\}$: then $A = B$ if and only if $B = C \diamond D$, if and only if $\phi A = \phi C \vee \phi D = \phi B$. $\square$

Corollary 7. *For all named sequents $\vdash \Gamma$ and renamings ϕ of $\vdash \Gamma$, $\phi(\vdash \Gamma)$ is also a named sequent with $(\vdash \Gamma) \equiv \phi(\vdash \Gamma)$, and sharing-free iff so is $\vdash \Gamma$.*

Proof. The fact that $\phi(\vdash \Gamma)$ is a named sequent and sharing-free iff so is $\vdash \Gamma$ follows from proposition 10 and lemma 21, point (ii). For the equivalence, observe that by lemma 21, point (i), ϕ induces a bijection between Γ and $\phi\Gamma$ such that (again by proposition 10) $A \equiv \phi A$ for all $A \in \Gamma$. $\square$

Finally, we can rename whole derivation trees while preserving their structure and conclusion (obviously up to renaming):

Definition 27. *A* renaming *of a named derivation tree $P \in \mathtt{GS4}^{\mathcal{N}}$ is an injective map $\phi : \mathtt{names}(P) \to \mathcal{N}$.*

The application of ϕ to P, noted ϕP, is the tree obtained by applying ϕ recursively to the named sequents labeling each node of P (plus the selected formulas in axiom rules).

Proposition 11. *Let $P \in \mathtt{GS4}^{\mathcal{N}}$ be any named derivation tree with conclusion $\vdash \Gamma$, ϕ a renaming of P: then $\phi P \in \mathtt{GS4}^{\mathcal{N}}$ is also a derivation tree with conclusion $\phi(\vdash \Gamma)$ and height $\mathtt{h}(\phi P) = \mathtt{h}(P)$, cut-free (resp. superposition-free) iff so is P.*

Proof. By a simple structural induction on P, using proposition 10 and corollary 7 to prove that constraints on the shape of rules are satisfied by ϕP at every step. $\square$

E Proofs of lemmas regarding simple axiom graphs

Proof of proposition 1. By structural induction on P:

- if P has the form
$$\frac{}{\vdash \Delta, A, \overline{B}}\ \mathtt{ax}_{\{A,\overline{B}\}}$$
where $\Gamma = \Delta \cup \{A, \overline{B}\}$ and $A \equiv B$ then by definitions 6 and 9 $V_{[\![P]\!]} = \mathtt{names}(\Delta) \cup \mathtt{names}(\{A, \overline{B}\}) = \mathtt{names}(\Gamma)$;
- if P has the form
$$\frac{\begin{array}{c}\vdots\ Q \\ \vdash \Gamma, A\end{array} \quad \begin{array}{c}\vdots\ R \\ \vdash \Gamma, \overline{A}\end{array}}{\vdash \Gamma}\ \mathtt{cut}$$
then by induction hypothesis and definition 8
$$V_{[\![P]\!]} = (\mathtt{names}(\Gamma, A) \cup \mathtt{names}(\Gamma, \overline{A})) \setminus \mathtt{names}(A);$$
the statement follows from the fact that $\mathtt{names}(A) = \mathtt{names}(\overline{A})$;

all other cases follow immediately from the induction hypothesis. $\square$

Proof of lemma 5. By structural induction on P. For the sake of brevity we let $P' = \mathtt{inv}(P, A \vee B)$ and omit at each step the shape of P, P': the interested reader may inspect the corresponding steps in the proof of lemma 1 (appendix C). Because $\mathtt{names}(\Gamma, A, B) = \mathtt{names}(\Gamma, A \vee B)$, we have in any case $V_{[\![P']\!]} = V_{[\![P]\!]}$ and it is enough to show that $E_{[\![P']\!]} = E_{[\![P]\!]}$:

- if P ends with an axiom rule application of the kind $\mathtt{ax}_{\{\overline{C} \wedge \overline{D}, A \vee B\}}$ with conclusion $\vdash \Delta, \overline{C} \wedge \overline{D}, A \vee B$ where $\Gamma = \Delta \cup \{\overline{C} \wedge \overline{D}\}$, then by definition 9
$$[\![P']\!] = (\mathtt{Wk}_{\Delta,B} \sqcup \mathtt{Id}_{\{\overline{C},A\}}) \sqcup (\mathtt{Wk}_{\Delta,A} \sqcup \mathtt{Id}_{\{\overline{D},B\}})$$
and by definition 6 we have
$$\begin{aligned}[\![P']\!] &= \mathtt{Wk}_{\Delta} \sqcup \mathtt{Id}_{\{\overline{C},A\}} \sqcup \mathtt{Id}_{\{\overline{D},B\}} \\ &= \mathtt{Wk}_{\Delta} \sqcup \mathtt{Id}_{\{\overline{C} \wedge \overline{D}, A \vee B\}} = [\![P]\!];\end{aligned}$$

– if P ends with an axiom rule application of the kind $\mathtt{ax}_{\{C,\overline{D}\}}$ with conclusion $\vdash \Delta, A \vee B, C, \overline{D}$ where $\Gamma = \Delta \cup \{C, \overline{D}\}$, then by definitions 6 and 9

$$[\![P']\!] = \mathtt{Wk}_{\Delta,A,B} \sqcup \mathtt{Id}_{\{C,\overline{D}\}} = \mathtt{Wk}_{\Delta,A\vee B} \sqcup \mathtt{Id}_{\{C,\overline{D}\}} = [\![P]\!];$$

– if P ends with a $\vee$-rule application introducing $A \vee B$ with premiss subtree Q, then $P' = Q$ and $[\![P']\!] = [\![Q]\!] = [\![P]\!]$;
– if P ends with a $\mathtt{cut}$-rule application on formulas $C, \overline{C}$, with premiss subderivations Q, R, we have by induction hypothesis $[\![\mathtt{inv}(Q, A \vee B)]\!] = [\![Q]\!]$ and $[\![\mathtt{inv}(R, A \vee B)]\!] = [\![R]\!]$, hence

$$[\![P']\!] = [\![\mathtt{inv}(Q, A \vee B)]\!] \odot_C [\![\mathtt{inv}(R, A \vee B)]\!] = [\![Q]\!] \odot_C [\![R]\!] = [\![P]\!];$$

– if P ends with a $\mathtt{sum}$-, $\vee$- or $\wedge$-rule application that does not introduce $A \vee B$, with premiss subtrees $Q_1, \ldots, Q_n$, we have $[\![\mathtt{inv}(Q_i, A \vee B)]\!] = [\![Q_i]\!]$ by induction hypothesis for all $1 \leq i \leq n$: then

$$\begin{aligned}[\![P']\!] = [\![\mathtt{inv}(Q_1, A \vee B)]\!] \sqcup \ldots \sqcup [\![\mathtt{inv}(Q_n, A \vee B)]\!] \\ = [\![Q_1]\!] \sqcup \ldots \sqcup [\![Q_n]\!] = [\![P]\!].\end{aligned}$$

□

Proof of lemma 6. We only prove the first part of the statement, i.e. that

$$[\![\mathtt{inv_1}(P, A \wedge B)]\!] \sqsubseteq [\![P]\!]\restriction_{\Gamma,A};$$

the argument for the second part is analogous. Let $P' = \mathtt{inv_1}(P, A \wedge B)$, As in the previous proof, we do not recall the shape of P, P'; the reader is invited to check our statements against the proofs of lemmas 2 and 3 (appendix C). Observe first that by proposition 1 we have

$$V_{[\![P']\!]} = \mathtt{names}(\Gamma, A) = V_{[\![P]\!]\restriction_{\Gamma,A}}.$$

We then need to prove only that $E_{[\![P']\!]} \subseteq E_{[\![P]\!]}$, as this will be enough to guarantee that $E_{[\![P']\!]} \subseteq E_{[\![P]\!]\restriction_{\Gamma,A}}$. We proceed by structural induction on P:

– if P ends with an axiom rule application of the kind $\mathtt{ax}_{\{\overline{C}\vee\overline{D},A\wedge B\}}$ with conclusion $\vdash \Delta, \overline{C} \vee \overline{D}, A \wedge B$ where $\Gamma = \Delta \cup \{\overline{C} \vee \overline{D}\}$, then by definitions 6 and 9

$$\begin{aligned}[\![P']\!] = \mathtt{Wk}_{\Delta,\overline{D}} \sqcup \mathtt{Id}_{\{\overline{C},A\}} \\ \sqsubseteq \mathtt{Wk}_{\Delta} \sqcup \mathtt{Id}_{\{\overline{C},A\}} \sqcup \mathtt{Id}_{\{\overline{D},B\}} = \mathtt{Wk}_{\Delta} \sqcup \mathtt{Id}_{\{\overline{C}\vee\overline{D},A\wedge B\}} = [\![P]\!];\end{aligned}$$

– if P ends with an axiom rule application of the kind $\mathtt{ax}_{\{C,\overline{D}\}}$ with conclusion $\vdash \Delta, A \wedge B, C, \overline{D}$ where $\Gamma = \Delta \cup \{C, \overline{D}\}$, then by definitions 6 and 9

$$[\![P']\!] = \mathtt{Wk}_{\Delta,A} \sqcup \mathtt{Id}_{\{C,\overline{D}\}} \sqsubseteq \mathtt{Wk}_{\Delta,A\wedge B} \sqcup \mathtt{Id}_{\{C,\overline{D}\}} = [\![P]\!];$$

– if P ends with a $\wedge$-rule application introducing $A \wedge B$ whose premiss subtrees Q, R have conclusions respectively $\vdash \Gamma, A$ and $\vdash \Gamma, B$, then $P' = Q$ and the conclusion is immediate;
– if P ends with a $\texttt{cut}$-rule application on formulas $C, \overline{C}$, with premiss subderivations Q, R, we have by induction hypothesis $[\![\texttt{inv}_\texttt{l}(Q, A \wedge B)]\!] \sqsubseteq [\![Q]\!]$ and $[\![\texttt{inv}_\texttt{l}(R, A \wedge B)]\!] \sqsubseteq [\![R]\!]$, with

$$[\![P']\!] = [\![\texttt{inv}_\texttt{l}(Q, A \wedge B)]\!] \odot_C [\![\texttt{inv}_\texttt{l}(R, A \wedge B)]\!].$$

By definition 8, $xy \in E_{[\![P']\!]}$ if and only if there is an alternating path (definition 7) connecting x with y between $[\![\texttt{inv}_\texttt{l}(Q, A \wedge B)]\!]$ and $[\![\texttt{inv}_\texttt{l}(R, A \wedge B)]\!]$ on inteface $\texttt{names}(C)$. It is easy to check that the same sequence of vertices is an alternating path between $[\![Q]\!]$ and $[\![R]\!]$ through interface $\texttt{names}(C)$, and therefore $xy \in E_{[\![Q]\!] \odot_C [\![R]\!]} = E_{[\![P]\!]}$;
– if P ends with a $\texttt{sum}$-, $\vee$- or $\wedge$-rule application that does not introduce $A \wedge B$, we conclude by induction hypothesis and proposition 7. □

Proof of proposition 2. Immediate by lemma 5 when A is a disjunction; for conjunctions we need to show that

$$[\![P]\!] = [\![\texttt{inv}_\texttt{l}(P, A)]\!] \sqcup [\![\texttt{inv}_\texttt{r}(P, A)]\!],$$

whenever P is cut-free. In particular, it suffices to show that

$$E_{[\![P]\!]} = E_{[\![\texttt{inv}_\texttt{l}(P,A)]\!]} \cup E_{[\![\texttt{inv}_\texttt{r}(P,A)]\!]}.$$

By structural induction on P, assuming $A = A_1 \wedge A_2$:

– if P ends with an axiom rule application of the kind $\texttt{ax}_{\{\overline{B}_1 \vee \overline{B}_2, A\}}$ with conclusion $\vdash \Delta, \overline{B}_1 \vee \overline{B}_2, A$ where $\Gamma = \Delta \cup \{\overline{B}_1 \vee \overline{B}_2\}$, then

$$\begin{aligned}[\![P]\!] = \texttt{Wk}_\Delta \sqcup \texttt{Id}_{\{\overline{B}_1 \vee \overline{B}_2, A\}} &= \texttt{Wk}_\Delta \sqcup \texttt{Id}_{\{\overline{B}_1, A_1\}} \sqcup \texttt{Id}_{\{\overline{B}_2, A_2\}},\\ [\![\texttt{inv}_\texttt{l}(P, A)]\!] &= \texttt{Wk}_{\Delta, \overline{B}_2} \sqcup \texttt{Id}_{\{\overline{B}_1, A_1\}},\\ [\![\texttt{inv}_\texttt{r}(P, A)]\!] &= \texttt{Wk}_{\Delta, \overline{B}_1} \sqcup \texttt{Id}_{\{\overline{B}_2, A_2\}}.\end{aligned}$$

Because the weakening graphs have no edges, we can conclude that

$$E_{[\![P]\!]} = E_{\texttt{Id}_{\{\overline{B}_1, A_1\}}} \sqcup E_{\texttt{Id}_{\{\overline{B}_2, A_2\}}} = E_{[\![\texttt{inv}_\texttt{l}(P,A)]\!]} \sqcup E_{[\![\texttt{inv}_\texttt{r}(P,A)]\!]}.$$

– if P ends with an axiom rule application of the kind $\texttt{ax}_{\{B, \overline{C}\}}$ with conclusion $\vdash \Delta, A, B, \overline{C}$ where $\Gamma = \Delta \cup \{B, \overline{C}\}$, then we have

$$\begin{aligned}[\![P]\!] &= \texttt{Wk}_\Delta \sqcup \texttt{Id}_{\{B, \overline{C}\}},\\ [\![\texttt{inv}_\texttt{l}(P, A)]\!] &= \texttt{Wk}_{\Delta, A_1} \sqcup \texttt{Id}_{\{B, \overline{C}\}},\\ [\![\texttt{inv}_\texttt{r}(P, A)]\!] &= \texttt{Wk}_{\Delta, A_2} \sqcup \texttt{Id}_{\{B, \overline{C}\}}.\end{aligned}$$

Therefore, because the weakening graphs have no edges,

$$E_{[\![P]\!]} = E_{\texttt{Id}_{\{B, \overline{C}\}}} = E_{[\![\texttt{inv}_\texttt{l}(P,A)]\!]} = E_{[\![\texttt{inv}_\texttt{r}(P,A)]\!]},$$

from which the conclusion follows.

– if P ends with a $\wedge$-rule application introducing A, the conclusion is trivial as $P = \mathtt{isl}(P, A)$;
– if P ends with a $\sqcup$-, $\vee$- or $\wedge$-rule application that does not introduce A, with premiss subtrees $Q_1, \ldots, Q_n$, we have by induction hypothesis

$$E_{[\![Q_i]\!]} = E_{[\![\mathtt{inv_l}(Q_i,A)]\!]} \cup E_{[\![\mathtt{inv_r}(Q_i,A)]\!]}.$$

for all $1 \leq i \leq n$. The result then follows from the fact that

$$E_{[\![P]\!]} = \bigcup_{i=1}^{n} E_{[\![Q_i]\!]} = \left(\bigcup_{i=1}^{n} E_{[\![\mathtt{inv_l}(Q_i,A)]\!]}\right) \cup \left(\bigcup_{i=1}^{n} E_{[\![\mathtt{inv_r}(Q_i,A)]\!]}\right). \qquad \square$$

F Proofs of lemmas regarding branch-labeled axiom graphs

F.1 Properties of branch sets

Lemma 22. *For all named formulas A, if $X \in \mathtt{Br}(A)$ then $X \subseteq \mathtt{names}(A)$.*

Proof. Immediate by induction on A. $\square$

Proof of lemma 7. Observe first that $\mathtt{names}(\Gamma) = \bigcup_{A\in\Gamma} \mathtt{names}(A)$, and assume $X \in \mathtt{Br}(\Gamma)$. By definition $X \subseteq \mathtt{names}(\Gamma)$, therefore $X = \bigcup_{A\in\Gamma}(X \cap \mathtt{names}(A))$; we know also that $(X \cap \mathtt{names}(A)) \in \mathtt{Br}(A)$ for all $A \in \Gamma$, hence we can choose $X_A = X \cap \mathtt{names}(A)$.

Conversely, assume $X = \bigcup_{A\in\Gamma} X_A$ with $X_A \in \mathtt{Br}(A)$ for all $A \in \Gamma$. By lemma 22, $X \subseteq \mathtt{names}(A)$; moreover, because Γ is sharing-free, its elements have pairwise disjoint name sets, hence the X_A in particular are pairwise disjoint and $X \cap \mathtt{names}(A) = X_A \in \mathtt{Br}(A)$, as desired. $\square$

Proof of lemma 8. Assume $X \in \mathtt{Br}(\Gamma \cup \Delta)$: by lemma 7 there is an indexed family $(X_A)_{A\in\Gamma\cup\Delta}$ of name sets such that $X = \bigcup_{A\in\Gamma\cup\Delta} X_A$ and $X_A \in \mathtt{Br}(A)$ for all $A \in \Gamma \cup \Delta$. Let then $Y = \bigcup_{A\in\Gamma} X_A$ and $Z = \bigcup_{A\in\Delta} X_A$: again by lemma 7, $Y \in \mathtt{Br}(\Gamma)$ and $Z \in \mathtt{Br}(\Delta)$, and obviously $X = Y \cup Z$. The argument for the converse is analogous, reversing the direction of all implications. $\square$

Proof of proposition 3. Fact (i) is left to the reader. For fact (ii), observe that $\mathtt{Br}(\{A, B\}) = \mathtt{Br}(\{A\} \cup \{B\}) = \mathtt{Br}(A \vee B)$ as an immediate consequence of lemma 8: then we have

$$\begin{aligned}\mathtt{Br}(\Gamma, A \vee B) &= \{X \cup Y \mid X \in \mathtt{Br}(\Gamma), Y \in \mathtt{Br}(A \vee B)\}\\ &= \{X \cup Y \mid X \in \mathtt{Br}(\Gamma), Y \in \mathtt{Br}(\{A, B\})\}\\ &= \mathtt{Br}(\Gamma, A, B).\end{aligned}$$

For fact (iii), recall that $\mathtt{Br}(A \wedge B) = \mathtt{Br}(A) \cup \mathtt{Br}(B)$; then, again by lemma 8,

$$\begin{aligned}
\mathtt{Br}(\Gamma, A \wedge B) &= \{X \cup Y \mid X \in \mathtt{Br}(\Gamma), Y \in \mathtt{Br}(A \wedge B)\} \\
&= \{X \cup Y \mid X \in \mathtt{Br}(\Gamma), Y \in \mathtt{Br}(A) \cup \mathtt{Br}(B)\} \\
&= \{X \cup Y \mid X \in \mathtt{Br}(\Gamma), Y \in \mathtt{Br}(A)\} \\
&\qquad \cup \{X \cup Y \mid X \in \mathtt{Br}(\Gamma), Y \in \mathtt{Br}(B)\} \\
&= \mathtt{Br}(\Gamma, A) \cup \mathtt{Br}(\Gamma, B).
\end{aligned}$$

For fact (iv), observe that (by an easy inductive argument) all branches of A (resp. of B) must be non-empty, and remember that Γ, A, B have disjoint name sets by hypothesis. Let then $X \in \mathtt{Br}(\Gamma, A) \cap \mathtt{Br}(\Gamma, B)$: by lemma 8 there is a *non-empty* $Y \in \mathtt{Br}(A)$ such that $Y \subseteq X$. By construction we have also $X \subseteq \mathtt{names}(\Gamma, B)$, but then there should be some name $z \in Y \subseteq \mathtt{names}(A)$ such that $z \in \mathtt{names}(\Gamma, B)$, which is impossible. □

F.2 Properties of branch-labeled axiom graphs

Lemma 23. *For all bl-graphs G,*

$$\mathtt{Br}(\mathtt{Wk}^{\mathtt{bl}}_{\Gamma}(G)) = \{X \cup Y \mid X \in \mathtt{Br}(G), Y \in \mathtt{Br}(\Gamma)\}.$$

Proof. Immediate, see definition 14. □

Lemma 24. *For any pair $A, \overline{B}$ of disjoint and sharing-free named formulas such that $A \equiv B$, $\mathtt{Br}(\mathtt{Id}^{\mathtt{bl}}_{\{A,\overline{B}\}}) = \mathtt{Br}(A, \overline{B})$.*

Proof. By induction on the height of A:

- if A is atomic, then $\mathtt{Br}(A, \overline{B}) = \{\mathtt{names}(A, \overline{B})\} = \mathtt{Br}(\mathtt{Id}^{\mathtt{bl}}_{\{A,\overline{B}\}})$ by proposition 3 and definition 14;
- if $A = A_1 \vee A_2$ and $\overline{B} = \overline{B}_1 \wedge \overline{B}_2$, then by definition 14

$$\begin{aligned}
\mathtt{Br}(\mathtt{Id}^{\mathtt{bl}}_{\{A,\overline{B}\}}) &= \mathtt{Br}(\mathtt{Wk}^{\mathtt{bl}}_{A_2}(\mathtt{Id}^{\mathtt{bl}}_{\{A_1,\overline{B}_1\}}) \sqcup \mathtt{Wk}^{\mathtt{bl}}_{A_1}(\mathtt{Id}^{\mathtt{bl}}_{\{A_2,\overline{B}_2\}})) \\
&= \mathtt{Br}(\mathtt{Wk}^{\mathtt{bl}}_{A_2}(\mathtt{Id}^{\mathtt{bl}}_{\{A_1,\overline{B}_1\}})) \cup \mathtt{Br}(\mathtt{Wk}^{\mathtt{bl}}_{A_1}(\mathtt{Id}^{\mathtt{bl}}_{\{A_2,\overline{B}_2\}})).
\end{aligned}$$

  By induction hypothesis, $\mathtt{Br}(\mathtt{Id}^{\mathtt{bl}}_{\{A_1,\overline{B}_1\}}) = \mathtt{Br}(A_1, \overline{B}_1)$ and $\mathtt{Br}(\mathtt{Id}^{\mathtt{bl}}_{\{A_2,\overline{B}_2\}}) = \mathtt{Br}(A_2, \overline{B}_2)$; by lemmas 8 and 23

$$\begin{aligned}
&\mathtt{Br}(\mathtt{Wk}^{\mathtt{bl}}_{A_2}(\mathtt{Id}^{\mathtt{bl}}_{\{A_1,\overline{B}_1\}})) \\
&\quad = \{X \cup Y \mid X \in \mathtt{Br}(A_1, \overline{B}_1), Y \in \mathtt{Br}(A_2)\} = \mathtt{Br}(A_1, A_2, \overline{B}_1), \\
&\mathtt{Br}(\mathtt{Wk}^{\mathtt{bl}}_{A_1}(\mathtt{Id}^{\mathtt{bl}}_{\{A_2,\overline{B}_2\}})) \\
&\quad = \{X \cup Y \mid X \in \mathtt{Br}(A_2, \overline{B}_2), Y \in \mathtt{Br}(A_1)\} = \mathtt{Br}(A_1, A_2, \overline{B}_2).
\end{aligned}$$

  Then, again by proposition 3,

$$\begin{aligned}
\mathtt{Br}(\mathtt{Id}^{\mathtt{bl}}_{\{A,\overline{B}\}}) &= \mathtt{Br}(A_1, A_2, \overline{B}_1) \cup \mathtt{Br}(A_1, A_2, \overline{B}_2) \\
&= \mathtt{Br}(A_1, A_2, \overline{B}_1 \wedge \overline{B}_2) \\
&= \mathtt{Br}(A_1 \vee A_2, \overline{B}_1 \wedge \overline{B}_2) = \mathtt{Br}(A, \overline{B}).
\end{aligned}$$

□

Corollary 8. *Let Γ be any sharing-free set of named formulas, $A, \overline{B}$ a pair of disjoint and sharing free named formulas that share no names with Γ and such that $A \equiv B$: then*

$$\mathtt{Br}(\mathtt{Wk}^{\mathtt{bl}}_{\Gamma}(\mathtt{Id}^{\mathtt{bl}}_{\{A,\overline{B}\}})) = \mathtt{Br}(\Gamma, A, \overline{B}).$$

Proof of proposition 4. The proof that $V_{(\!|P|\!)} = \mathtt{names}(\Gamma)$ is analogous to that of proposition 1 and left to the reader. We prove by structural induction on P that $\mathtt{Br}((\!|P|\!)) \subseteq \mathtt{Br}(\Gamma)$:

- if P is an axiom rule application, the conclusion follows immediately from corollary 8;
- if P has the form

$$\dfrac{\begin{array}{cc}\vdots\, Q & \vdots\, R \\ \vdash \Gamma, A & \vdash \Gamma, \overline{A}\end{array}}{\vdash \Gamma}\,\mathtt{cut}$$

  then by definition 13, if $X \in \mathtt{Br}((\!|Q|\!) \odot_A (\!|R|\!))$ there is either $Y \in \mathtt{Br}((\!|Q|\!))$ or $Y \in \mathtt{Br}((\!|R|\!))$ such that $X = Y \setminus \mathtt{names}(A)$. By induction hypothesis we have either $Y \in \mathtt{Br}(\Gamma, A)$ or $Y \in \mathtt{Br}(\Gamma, \overline{A})$, hence $X \in \mathtt{Br}(\Gamma)$ by corollary 3;

all other cases follow immediately from the induction hypothesis. □

F.3 Behaviour under inversion

Lemma 25. *For all bl-graphs G, H and sharing-free sets Γ of named formulas,*

$$\mathtt{Wk}^{\mathtt{bl}}_{\Gamma}(G \sqcup H) = \mathtt{Wk}^{\mathtt{bl}}_{\Gamma}(G) \sqcup \mathtt{Wk}^{\mathtt{bl}}_{\Gamma}(H).$$

Proof. Left to the reader. □

Lemma 26. *For all bl-graphs G and sharing-free sets $\Gamma \cup \Delta$ of named formulas,*

$$\mathtt{Wk}^{\mathtt{bl}}_{\Gamma}(\mathtt{Wk}^{\mathtt{bl}}_{\Delta}(G)) = \mathtt{Wk}^{\mathtt{bl}}_{\Gamma \cup \Delta}(G).$$

Proof. The identity is obvious for vertices. For the edge-branch relation, let $e \prec_{\mathtt{Wk}^{\mathtt{bl}}_{\Gamma}(\mathtt{Wk}^{\mathtt{bl}}_{\Delta}(G))} X$: there are then $Y \in \mathtt{Br}(G)$, $Z \in \mathtt{Br}(\Delta)$ and $W \in \mathtt{Br}(\Gamma)$ such that $e \prec_G Y$ and $X = Y \cup Z \cup W$. By lemma 8, $Z \cup W \in \mathtt{Br}(\Gamma \cup \Delta)$, hence $e \prec_{\mathtt{Wk}^{\mathtt{bl}}_{\Gamma \cup \Delta}(G)} X$. Conversely, let $e \prec_{\mathtt{Wk}^{\mathtt{bl}}_{\Gamma \cup \Delta}(G)} X$: there are $Y \in \mathtt{Br}(G)$, $Z \in \mathtt{Br}(\Gamma \cup \Delta)$ such that $e \prec_G Y$ and $X = Y \cup Z$. Again by lemma 8, $Z = Z' \cup Z''$ for some $Z' \in \mathtt{Br}(\Gamma)$, $Z'' \in \mathtt{Br}(\Delta)$, hence $e \prec_{\mathtt{Wk}^{\mathtt{bl}}_{\Gamma}(\mathtt{Wk}^{\mathtt{bl}}_{\Delta}(G))} X$. □

Proof of lemma 10. Almost identical to the proof of lemma 5 (appendix E); only the base case is different (axioms). Let P' denote the inverted derivation $\mathtt{inv}(P, A \vee B)$; then:

– if P ends with an axiom rule application of the kind $\mathtt{ax}_{\{\overline{C}\wedge\overline{D},A\vee B\}}$ with conclusion $\vdash \Delta, \overline{C}\wedge\overline{D}, A\vee B$ where $\Gamma = \Delta \cup \{\overline{C}\wedge\overline{D}\}$, then by definition 15 we have

$$(\!|P|\!) = \mathtt{Wk}^{\mathtt{bl}}_{\Delta}(\mathtt{Id}^{\mathtt{bl}}_{\{\overline{C}\wedge\overline{D},A\vee B\}}),$$
$$(\!|P'|\!) = \mathtt{Wk}^{\mathtt{bl}}_{\Delta,B}(\mathtt{Id}^{\mathtt{bl}}_{\{\overline{C},A\}}) \sqcup \mathtt{Wk}^{\mathtt{bl}}_{\Delta,A}(\mathtt{Id}^{\mathtt{bl}}_{\{\overline{D},B\}}),$$

and by definition 14

$$(\!|P|\!) = \mathtt{Wk}^{\mathtt{bl}}_{\Delta}(\mathtt{Wk}^{\mathtt{bl}}_{B}(\mathtt{Id}^{\mathtt{bl}}_{\{\overline{C},A\}}) \sqcup \mathtt{Wk}^{\mathtt{bl}}_{A}(\mathtt{Id}^{\mathtt{bl}}_{\{\overline{D},B\}})).$$

We apply lemmas 25 and 26 to get

$$\begin{aligned}(\!|P|\!) &= \mathtt{Wk}^{\mathtt{bl}}_{\Delta}(\mathtt{Wk}^{\mathtt{bl}}_{B}(\mathtt{Id}^{\mathtt{bl}}_{\{\overline{C},A\}})) \sqcup \mathtt{Wk}^{\mathtt{bl}}_{\Delta}(\mathtt{Wk}^{\mathtt{bl}}_{A}(\mathtt{Id}^{\mathtt{bl}}_{\{\overline{D},B\}}))\\ &= \mathtt{Wk}^{\mathtt{bl}}_{\Delta,B}(\mathtt{Id}^{\mathtt{bl}}_{\{\overline{C},A\}}) \sqcup \mathtt{Wk}^{\mathtt{bl}}_{\Delta,A}(\mathtt{Id}^{\mathtt{bl}}_{\{\overline{D},B\}}) = (\!|P'|\!).\end{aligned}$$

– if P ends with an axiom rule application of the kind $\mathtt{ax}_{\{C,\overline{D}\}}$ with conclusion $\vdash \Delta, A\vee B, C, \overline{D}$ where $\Gamma = \Delta\cup\{C,\overline{D}\}$, then by definition 15 and proposition 3

$$(\!|P'|\!) = \mathtt{Wk}^{\mathtt{bl}}_{\Delta,A,B}(\mathtt{Id}^{\mathtt{bl}}_{\{C,\overline{D}\}}) = \mathtt{Wk}^{\mathtt{bl}}_{\Delta,A\vee B}(\mathtt{Id}^{\mathtt{bl}}_{\{C,\overline{D}\}}) = (\!|P|\!). \qquad \square$$

Lemma 27. *For all bl-graphs $G \sqsubseteq H$ and name sets $X \subseteq \mathcal{N}$,*

$$V_G \subseteq X \implies G \sqsubseteq H\restriction_X.$$

Proof. If $V_G \subseteq X$ then obviously $V_G \subseteq V_{H\restriction_X} = (V_H \cap X)$. Moreover, let $e \prec_G Y$: we have $Y \subseteq V_G \subseteq X$ by definition 11 and we know by hypothesis that $e \prec_H Y$, hence $e \prec_{H\restriction_X} Y$. $\square$

Lemma 28. *Let $\Gamma \cup \{A \wedge B\}$ be a sharing-free set of named formulas, G any bl-graph with $\mathtt{Br}(G) \subseteq \mathtt{Br}(\Gamma, A\wedge B)$: then*

$$\mathtt{Br}(G\restriction_{\Gamma,A}) \subseteq \mathtt{Br}(\Gamma, A) \text{ and } \mathtt{Br}(G\restriction_{\Gamma,B}) \subseteq \mathtt{Br}(\Gamma, B).$$

Proof. We only prove the first half of the statement, the second half is similar. Observe that (by an easy inductive argument) every branch of B is non-empty, and remember that Γ, A, B have disjoint name sets. Let $X \in \mathtt{Br}(G\restriction_{\Gamma,A})$ and assume $X \in \mathtt{Br}(\Gamma, B)$: by lemma 8 there are $Y \in \mathtt{Br}(\Gamma)$ and a *non-empty* $Z \in \mathtt{Br}(B)$ such that $X = Y \cup Z$. However, we have $X \subseteq \mathtt{names}(\Gamma, A)$ by construction, hence $Z \subseteq \mathtt{names}(\Gamma, A)$, which is disjoint from $\mathtt{names}(B)$, a contradiction. Therefore $X \notin \mathtt{Br}(\Gamma, B)$, and by exclusion we have $X \in \mathtt{Br}(\Gamma, A)$. $\square$

Proof of lemma 11. We only prove the first part of the statement, i.e. that

$$(\!|\mathtt{inv}_1(P, A\wedge B)|\!) = (\!|P|\!)\restriction_{\Gamma,A};$$

the argument for the second part is analogous. Let $P' = \mathtt{inv}_1(P, A\wedge B)$, As usual we reason about the shape of P, P' without recalling it; the interested reader may inspect the proofs of lemmas 2 and 3 (appendix C). Observe first that by proposition 4 we have

$$V_{(\!|P'|\!)} = \mathtt{names}(\Gamma, A) = V_{(\!|P|\!)\restriction_{\Gamma,A}};$$

by lemma 27, then, we can prove that $(\!|P'|\!) \sqsubseteq (\!|P|\!)\restriction_{\Gamma,A}$ by showing that $(\!|P'|\!) \sqsubseteq (\!|P|\!)$. For the reverse inclusion, on the other hand, it suffices to show that $\prec_{(\!|P|\!)\restriction_{\Gamma,A}} \subseteq \prec_{(\!|P'|\!)}$. By structural induction on P:

- if P ends with an axiom rule application of the kind $\mathtt{ax}_{\{\overline{C}\vee\overline{D},A\wedge B\}}$ with conclusion $\vdash \Delta, \overline{C} \vee \overline{D}, A \wedge B$ where $\Gamma = \Delta \cup \{\overline{C} \vee \overline{D}\}$, then by definition 15

  $$(\!|P|\!) = \mathtt{Wk}^{\mathtt{bl}}_{\Delta}(\mathtt{Id}^{\mathtt{bl}}_{\{\overline{C}\vee\overline{D},A\wedge B\}}) \text{ and } (\!|P'|\!) = \mathtt{Wk}^{\mathtt{bl}}_{\Delta,\overline{D}}(\mathtt{Id}^{\mathtt{bl}}_{\{\overline{C},A\}}).$$

  By definition 14 and lemmas 25 and 26

  $$\begin{aligned}(\!|P|\!) &= \mathtt{Wk}^{\mathtt{bl}}_{\Delta}(\mathtt{Wk}^{\mathtt{bl}}_{\overline{D}}(\mathtt{Id}^{\mathtt{bl}}_{\{\overline{C},A\}}) \sqcup \mathtt{Wk}^{\mathtt{bl}}_{\overline{C}}(\mathtt{Id}^{\mathtt{bl}}_{\{\overline{D},B\}}))\\ &= \mathtt{Wk}^{\mathtt{bl}}_{\Delta,\overline{D}}(\mathtt{Id}^{\mathtt{bl}}_{\{\overline{C},A\}}) \sqcup \mathtt{Wk}^{\mathtt{bl}}_{\Delta,\overline{C}}(\mathtt{Id}^{\mathtt{bl}}_{\{\overline{D},B\}})\\ &= (\!|P'|\!) \sqcup \mathtt{Wk}^{\mathtt{bl}}_{\Delta,\overline{C}}(\mathtt{Id}^{\mathtt{bl}}_{\{\overline{D},B\}}),\end{aligned}$$

  hence $(\!|P'|\!) \sqsubseteq (\!|P|\!)$, and we have also

  $$\mathtt{Br}(\mathtt{Wk}^{\mathtt{bl}}_{\Delta,\overline{C}}(\mathtt{Id}^{\mathtt{bl}}_{\{\overline{D},B\}})) = \mathtt{Br}(\Delta, \overline{C}, \overline{D}, B) = \mathtt{Br}(\Gamma, B)$$

  by corollary 8 and proposition 3. Let then $e \prec_{(\!|P|\!)\restriction_{\Gamma,A}} X$: by lemma 28 $X \in \mathtt{Br}(\Gamma, A)$, hence by proposition 3, point (iv)

  $$X \notin \mathtt{Br}(\Gamma, B) = \mathtt{Br}(\mathtt{Wk}^{\mathtt{bl}}_{\Delta,\overline{C}}(\mathtt{Id}^{\mathtt{bl}}_{\{\overline{D},B\}})),$$

  and then by exclusion we must have $e \prec_{(\!|P'|\!)} X$;
- if P ends with an axiom rule application of the kind $\mathtt{ax}_{\{C,\overline{D}\}}$ with conclusion $\vdash \Delta, A \wedge B, C, \overline{D}$ where $\Gamma = \Delta \cup \{C, \overline{D}\}$, then by definition 9

  $$(\!|P|\!) = \mathtt{Wk}^{\mathtt{bl}}_{\Delta,A\wedge B}(\mathtt{Id}^{\mathtt{bl}}_{\{C,\overline{D}\}}) \text{ and } (\!|P'|\!) = \mathtt{Wk}^{\mathtt{bl}}_{\Delta,A}(\mathtt{Id}^{\mathtt{bl}}_{\{C,\overline{D}\}}).$$

  It is easy to check, using definition 14, proposition 3, and corollary 8, that

  $$(\!|P|\!) = \mathtt{Wk}^{\mathtt{bl}}_{\Delta,A}(\mathtt{Id}^{\mathtt{bl}}_{\{C,\overline{D}\}}) \sqcup \mathtt{Wk}^{\mathtt{bl}}_{\Delta,B}(\mathtt{Id}^{\mathtt{bl}}_{\{C,\overline{D}\}}) = (\!|P'|\!) \sqcup \mathtt{Wk}^{\mathtt{bl}}_{\Delta,B}(\mathtt{Id}^{\mathtt{bl}}_{\{C,\overline{D}\}})$$

  with

  $$\mathtt{Br}(\mathtt{Wk}^{\mathtt{bl}}_{\Delta,B}(\mathtt{Id}^{\mathtt{bl}}_{\{C,\overline{D}\}})) = \mathtt{Br}(\Delta, B, C, \overline{D}) = \mathtt{Br}(\Gamma, B),$$

  and we conclude as in the previous case;
- if P ends with a $\wedge$-rule application introducing $A \wedge B$ whose premiss subtrees Q, R have conclusions respectively $\vdash \Gamma, A$ and $\vdash \Gamma, B$, then $P' = Q$. We have obviously $(\!|P'|\!) \sqsubseteq (\!|Q|\!) \sqcup (\!|R|\!) = (\!|P|\!)$. For the reverse inclusion, we argue by exclusion from lemma 28 and proposition 3 and the fact that $\mathtt{Br}((\!|R|\!)) \subseteq \mathtt{Br}(\Gamma, B)$, concluding that $\prec_{(\!|P|\!)\restriction_{\Gamma,A}} \subseteq \prec_{(\!|Q|\!)} = \prec_{(\!|P'|\!)}$;

– if P ends with a `cut`-rule application on formulas $C, \overline{C}$, with premiss subderivations Q, R, we have by induction hypothesis

$$(\!|\mathtt{inv_1}(Q, A \wedge B)|\!) = (\!|Q|\!)\restriction_{\Gamma,A,C} \text{ and } (\!|\mathtt{inv_1}(R, A \wedge B)|\!) = (\!|R|\!)\restriction_{\Gamma,A,\overline{C}},$$

hence

$$(\!|P'|\!) = (\!|Q|\!)\restriction_{\Gamma,A,C} \odot_C (\!|R|\!)\restriction_{\Gamma,A,\overline{C}}.$$

Let $e \prec_{(\!|P'|\!)} X$: there is by definition 13 an X-labeled alternating path $z_1, \ldots, z_n$ between $(\!|Q|\!)\restriction_{\Gamma,A,C}$ and $(\!|R|\!)\restriction_{\Gamma,A,\overline{C}}$ through interface $\mathtt{names}(C)$, with $e = z_1 z_n$. This is obviously also an X-labeled alternating path between $(\!|Q|\!)$ and $(\!|R|\!)$ through the same interface, hence $e \prec_{(\!|P|\!)} X$ and $(\!|P'|\!) \sqsubseteq (\!|P|\!)$.

Now let $e \prec_{(\!|P|\!)\restriction_{\Gamma,A}} X$: there is, again by definition 13, an X-labeled alternating path $z_1, \ldots, z_n$ between $(\!|Q|\!)$ and $(\!|R|\!)$ through interface $\mathtt{names}(C)$ with $e = z_1 z_n$. We have to show that this is also an X-labeled alternating path between $(\!|Q|\!)\restriction_{\Gamma,A,C}$ and $(\!|R|\!)\restriction_{\Gamma,A,\overline{C}}$ through the same interface, from which it follows that $e \prec_{(\!|P'|\!)} X$. Let $I = \mathtt{names}(C)$ and assume $z_i z_{i+1} \prec^I_{(\!|Q|\!)} X$ for some $1 \le i < n$: there is $Y \subseteq I$ such that $z_i z_{i+1} \prec_{(\!|Q|\!)} (X \cup Y)$; we know also that $X \subseteq \mathtt{names}(\Gamma, A)$, hence $(X \cup Y) \subseteq \mathtt{names}(\Gamma, A, C)$: then

$$z_i z_{i+1} \prec_{(\!|Q|\!)\restriction_{\Gamma,A,C}} (X \cup Y) \text{ and } z_i z_{i+1} \prec^I_{(\!|Q|\!)\restriction_{\Gamma,A,C}} X.$$

Similarly, if $z_i z_{i+1} \prec^I_{(\!|R|\!)} X$ then $z_i z_{i+1} \prec^I_{(\!|R|\!)\restriction_{\Gamma,A,\overline{C}}} X$, and we conclude as desired;

– if P ends with a `sum`-, $\vee$- or $\wedge$-rule application that does not introduce $A \wedge B$, we conclude by induction hypothesis and the fact that restriction distributes over graph unions. □

Proof of lemma 12. Observe first that

$$\mathtt{names}(\Gamma, A \wedge B) = \mathtt{names}(\Gamma, A) \cup \mathtt{names}(\Gamma, B),$$

hence

$$V_{(\!|P|\!)} = \mathtt{names}(\Gamma, A \wedge B) = V_{(\!|P|\!)\restriction_{\Gamma,A}} \cup V_{(\!|P|\!)\restriction_{\Gamma,B}}.$$

For the branch-label relation, we start by recalling the fact that

$$\mathtt{Br}((\!|P|\!)) \subseteq \mathtt{Br}(\Gamma, A \wedge B) = \mathtt{Br}(\Gamma, A) \cup \mathtt{Br}(\Gamma, B).$$

Now let $e \prec_{(\!|P|\!)} X$: if $X \in \mathtt{Br}(\Gamma, A)$, then in particular $X \subseteq \mathtt{names}(\Gamma, A)$, hence $e \prec_{(\!|P|\!)\restriction_{\Gamma,A}} X$; similarly, if $X \in \mathtt{Br}(\Gamma, B)$ then $e \prec_{(\!|P|\!)\restriction_{\Gamma,B}} X$. We have then

$$(\!|P|\!) \sqsubseteq (\!|P|\!)\restriction_{\Gamma,A} \sqcup (\!|P|\!)\restriction_{\Gamma,B};$$

the reverse inclusion is immediate. □

F.4 Auxiliary lemmas for cut-elimination

Proof of lemma 13. By structural induction on P; we leave the base case (axioms), superpositions and the logical rules to the reader. For cuts, assume P has the form

$$\dfrac{\begin{array}{cc} \vdots\, Q & \vdots\, R \\ \vdash \Gamma, A & \vdash \Gamma, \overline{A} \end{array}}{\vdash \Gamma}\,\texttt{cut}$$

We have then $(\!|P|\!) = (\!|Q|\!) \odot_A (\!|R|\!)$. Let $xy \in E_{(\!|P|\!)}$: there is by definition 13 a labeled alternating path $z_1, \ldots, z_n$ between $(\!|Q|\!)$ and $(\!|R|\!)$ through the interface $\texttt{names}(A)$, with $xy = z_1 z_n$. For all $1 < i < n$ (the interface nodes) we have $\overline{A}[z_i] = \overline{A[z_i]}$. Let $\alpha = \Gamma[x]$. There are two possibilities:

- *odd edges in $(\!|Q|\!)$, even edges in $(\!|R|\!)$:* we prove by secondary induction on i that for all $1 < i < n$, if i is odd then $A[z_i] = \alpha$, if it is even then $A[z_i] = \overline{\alpha}$. If $i = 2$ (even) then $z_1 z_2 \in E_{(\!|Q|\!)}$, hence by the primary induction hypothesis

  $$A[z_2] = (\Gamma, A)[z_2] = \overline{(\Gamma, A)[z_1]} = \overline{\Gamma[z_1]} = \overline{\Gamma[x]} = \overline{\alpha}.$$

  If $i > 2$ and even, then $z_{i-1} z_i \in E_{(\!|Q|\!)}$; by the secondary induction hypothesis $A[z_{i-1}] = \alpha$, and by the primary hypothesis

  $$A[z_i] = (\Gamma, A)[z_i] = \overline{(\Gamma, A)[z_{i-1}]} = \overline{A[z_{i-1}]} = \overline{\alpha}.$$

  If $i > 2$ and odd, then $z_{i-1} z_i \in E_{(\!|R|\!)}$; by the secondary induction hypothesis $A[z_{i-1}] = \overline{\alpha}$ hence $\overline{A}[z_{i-1}] = \alpha$; by the primary hypothesis

  $$\overline{A}[z_i] = (\Gamma, \overline{A})[z_i] = \overline{(\Gamma, \overline{A})[z_{i-1}]} = \overline{\overline{A}[z_{i-1}]} = \overline{\alpha},$$

  therefore $A[z_i] = \overline{\overline{\alpha}} = \alpha$. We conclude by cases on n: if $n = 2$, then by the primary induction hypothesis

  $$\Gamma[y] = (\Gamma, A)[z_2] = \overline{(\Gamma, A)[z_1]} = \overline{\Gamma[x]};$$

  if $n > 2$ and even, then $z_{n-1} z_n \in E_{(\!|Q|\!)}$ and $A[z_{n-1}] = \alpha$; by the primary induction hypothesis

  $$\Gamma[y] = (\Gamma, A)[z_n] = \overline{(\Gamma, A)[z_{n-1}]} = \overline{A[z_{n-1}]} = \overline{\alpha} = \overline{\Gamma[x]};$$

  if $n > 2$ and odd, then $z_{n-1} z_n \in E_{(\!|R|\!)}$ and $A[z_{n-1}] = \overline{\alpha}$, hence $\overline{A}[z_{n-1}] = \alpha$; by the primary induction hypothesis

  $$\Gamma[y] = (\Gamma, \overline{A})[z_n] = \overline{(\Gamma, \overline{A})[z_{n-1}]} = \overline{\overline{A}[z_{n-1}]} = \overline{\alpha} = \overline{\Gamma[x]}.$$

- *odd edges in $(\!|R|\!)$, even edges in $(\!|Q|\!)$:* we proceed as in the previous case, swapping Q with R and A with $\overline{A}$. ☐

Proof of lemma 16. By structural induction on P. We prove the result only for the base case (i.e. the axioms), all other cases are more or less immediate.

Case (i). P is an axiom rule application of the kind $\mathtt{ax}_{\{\overline{B},A\}}$ with conclusion $\vdash \Delta, \overline{B}, A$, where $\Gamma = \Delta \cup \{\overline{B}\}$:

– if $A = A_1 \vee A_2$ is a disjunction, then $\mathtt{inv}(P, A)$ has the form

$$\dfrac{\dfrac{\dfrac{}{\vdash \Delta, \overline{B}_1, A_1, A_2}\,\mathtt{ax}_{\{\overline{B}_1,A_1\}} \qquad \dfrac{}{\vdash \Delta, \overline{B}_2, A_1, A_2}\,\mathtt{ax}_{\{\overline{B}_2,A_2\}}}{\vdash \Delta, \overline{B}_1 \wedge \overline{B}_2, A_1, A_2}\,\wedge}{\vdash \Delta, \overline{B}_1 \wedge \overline{B}_2, A_1 \vee A_2}\,\vee$$

where $\overline{B} = \overline{B}_1 \wedge \overline{B}_2$. We have

$$\begin{aligned}
\mathtt{vh}(P) &= 1 + \mathtt{deg}(\vdash \Delta, \overline{B}_1 \wedge \overline{B}_2, A_1 \vee A_2)\\
&= 1 + \mathtt{deg}(\vdash \Delta)\\
&\qquad + 1 + \mathtt{deg}(\overline{B}_1) + \mathtt{deg}(\overline{B}_2)\\
&\qquad + 1 + \mathtt{deg}(A_1) + \mathtt{deg}(A_2)\\
&= 3 + \mathtt{deg}(\vdash \Delta, \overline{B}_1, \overline{B}_2, A_1, A_2)
\end{aligned}$$

and

$$\begin{aligned}
\mathtt{vh}(\mathtt{inv}(P, A)) &= 2 + \max\{1 + \mathtt{deg}(\vdash \Delta, \overline{B}_1, A_1, A_2), 1 + \mathtt{deg}(\vdash \Delta, \overline{B}_2, A_1, A_2)\}\\
&= 3 + \max\{\mathtt{deg}(\vdash \Delta, \overline{B}_1, A_1, A_2), \mathtt{deg}(\vdash \Delta, \overline{B}_2, A_1, A_2)\}.
\end{aligned}$$

Clearly,

$$\mathtt{deg}(\vdash \Delta, \overline{B}_1, A_1, A_2), \mathtt{deg}(\vdash \Delta, \overline{B}_2, A_1, A_2) \leq \mathtt{deg}(\vdash \Delta, \overline{B}_1, \overline{B}_2, A_1, A_2),$$

hence $\mathtt{vh}(\mathtt{inv}(P, A)) \leq \mathtt{vh}(P)$;

– if $A = A_1 \wedge A_2$ is a conjunction, then $\mathtt{inv}(P, A)$ has the form

$$\dfrac{\dfrac{\dfrac{}{\vdash \Delta, \overline{B}_1, \overline{B}_2, A_1}\,\mathtt{ax}_{\{\overline{B}_1,A_1\}}}{\vdash \Delta, \overline{B}_1 \vee \overline{B}_2, A_1}\,\vee \qquad \dfrac{\dfrac{}{\vdash \Delta, \overline{B}_1, \overline{B}_2, A_2}\,\mathtt{ax}_{\{\overline{B}_2,A_2\}}}{\vdash \Delta, \overline{B}_1 \vee \overline{B}_2, A_2}\,\vee}{\vdash \Delta, \overline{B}_1 \wedge \overline{B}_2, A_1 \vee A_2}\,\wedge$$

where $\overline{B} = \overline{B}_1 \vee \overline{B}_2$. We have as before

$$\mathtt{vh}(P) = 3 + \mathtt{deg}(\vdash \Delta, \overline{B}_1, \overline{B}_2, A_1, A_2)$$

and

$$\begin{aligned}
\mathtt{vh}(\mathtt{inv}(P, A)) &= 1 + \max\{2 + \mathtt{deg}(\vdash \Delta, \overline{B}_1, \overline{B}_2, A_1), 2 + \mathtt{deg}(\vdash \Delta, \overline{B}_1, \overline{B}_2, A_2)\}\\
&= 3 + \max\{\mathtt{deg}(\vdash \Delta, \overline{B}_1, \overline{B}_2, A_1), \mathtt{deg}(\vdash \Delta, \overline{B}_1, \overline{B}_2, A_2)\},
\end{aligned}$$

with

$$\mathtt{deg}(\vdash \Delta, \overline{B}_1, \overline{B}_2, A_1), \mathtt{deg}(\vdash \Delta, \overline{B}_1, \overline{B}_2, A_2) \leq \mathtt{deg}(\vdash \Delta, \overline{B}_1, \overline{B}_2, A_1, A_2),$$

and again $\mathtt{vh}(\mathtt{inv}(P, A)) \leq \mathtt{vh}(P)$.

Case (ii). P is an axiom rule application of the kind $\mathtt{ax}_{\{\overline{B},C\}}$ with conclusion $\vdash \Delta, A, \overline{B}, C$, where $\Gamma = \Delta \cup \{\overline{B}, C\}$:

– if $A = A_1 \vee A_2$ is a disjunction, then $\mathtt{inv}(P, A)$ has the form

$$\dfrac{\dfrac{}{\vdash \Delta, A_1, A_2, \overline{B}, C}\,\mathtt{ax}_{\{\overline{B},C\}}}{\vdash \Delta, A_1 \vee A_2, \overline{B}, C}\,\vee$$

We have

$$\begin{aligned}\mathtt{vh}(P) &= 1 + \mathtt{deg}(\vdash \Delta, A_1 \vee A_2, \overline{B}, C)\\ &= 1 + \mathtt{deg}(\vdash \Delta, \overline{B}, C)\\ &\quad\quad + 1 + \mathtt{deg}(A_1) + \mathtt{deg}(A_2)\\ &= 2 + \mathtt{deg}(\vdash \Delta, A_1, A_2, \overline{B}, C)\end{aligned}$$

and

$$\mathtt{vh}(\mathtt{inv}(P, A)) = 2 + \mathtt{deg}(\vdash \Delta, A_1, A_2, \overline{B}, C),$$

hence *a fortiori* $\mathtt{vh}(\mathtt{inv}(P, A)) \leq \mathtt{vh}(P)$;

– if $A = A_1 \wedge A_2$ is a conjunction, then $\mathtt{inv}(P, A)$ has the form

$$\dfrac{\dfrac{}{\vdash \Delta, A_1, \overline{B}, C}\,\mathtt{ax}_{\{\overline{B},C\}} \qquad \dfrac{}{\vdash \Delta, A_2, \overline{B}, C}\,\mathtt{ax}_{\{\overline{B},C\}}}{\vdash \Delta, A_1 \wedge A_2, \overline{B}, C}\,\wedge$$

We have as before

$$\mathtt{vh}(P) = 2 + \mathtt{deg}(\vdash \Delta, A_1, A_2, \overline{B}, C)$$

and

$$\begin{aligned}\mathtt{vh}(\mathtt{inv}(P, A)) &= 1 + \max\{1 + \mathtt{deg}(\vdash \Delta, A_1, \overline{B}, C), 1 + \mathtt{deg}(\vdash \Delta, A_2, \overline{B}, C)\}\\ &= 2 + \max\{\mathtt{deg}(\vdash \Delta, A_1, \overline{B}, C), \mathtt{deg}(\vdash \Delta, A_2, \overline{B}, C)\}\end{aligned}$$

with

$$\mathtt{deg}(\vdash \Delta, A_1, \overline{B}, C), \mathtt{deg}(\vdash \Delta, A_2, \overline{B}, C) \leq \mathtt{deg}(\vdash \Delta, A_1, A_2, \overline{B}, C)$$

hence $\mathtt{vh}(\mathtt{inv}(P, A)) \leq \mathtt{vh}(P)$. □

G Proof of the semantic cut-admissibility lemma

We devote the present section to the proof of lemma 14, which requires a somewhat involved construction. We start by fixing a set

$$\mathtt{Pol} = \{\circ, \bullet\}$$

of *polarities*. For $p \in \mathtt{Pol}$, we write $\overline{p}$ for the opposite polarity, i.e. let $\overline{\circ} = \bullet$ and $\overline{\bullet} = \circ$. As background data for the construction, we have to provide a pair of bl-graphs $G_\circ, G_\bullet$ and a *finite* set of names $I \subseteq \mathcal{N}$ to act as the composition interface. The two graphs are required to satisfy the following property:

there is a unique branch name $X \subseteq \mathcal{N}$ such that, for all $p \in \texttt{Pol}$ and $e \in E_{G_p}$, $e \prec^I_{G_p} X$;

in other words, the two graphs must have collectively a unique branch up to names in the interface. The restriction has two important consequences:

- all alternating paths between the two graphs through I are necessarily X-labeled, hence composable;
- any edge can be added to any alternating path, without caring for its label.

More formally, let $z_1, \ldots, z_n$ be an X-labeled alternating path between $G_\circ$ and $G_\bullet$ through interface I (definition 12). For any $p \in \texttt{Pol}$ we call the path

- *p-initial*, if all odd edges are in G_p and all even edges in $G_{\overline{p}}$;
- *p-final*, if it is p-initial with even n, or $\overline{p}$-initial with odd n.

Because the graphs and interface – as well as the unique branch label – are fixed, from now on we are going to speak simply of alternating paths, without mentioning the full expression (X-labeled alt. paths between $G_\circ$ and $G_\bullet$ through interface I). The following properties hold:

Lemma 29. *For all $p \in \texttt{Pol}$ and edge $xy \in E_{G_p}$, x, y is a p-initial and p-final alternating path.*

Lemma 30. *For any $p, q \in \texttt{Pol}$, if $z_1, \ldots, z_n$ is a p-initial and q-final alternating path then $z_n, \ldots, z_1$ is a q-initial and p-final alternating path.*

Lemma 31. *Let $p \in \texttt{Pol}$, and let $\boldsymbol{z} = z_1, \ldots, z_n$, $\boldsymbol{w} = w_1, \ldots, w_n$ be alternating paths such that $\boldsymbol{z}$ is p-final, $\boldsymbol{w}$ is $\overline{p}$-initial and $z_n = w_1$: then*

$$\boldsymbol{zw} = z_1, \ldots, (z_n = w_1), \ldots, w_n$$

is also an alternating path, q-initial iff so is $\boldsymbol{z}$ and q-final iff so is $\boldsymbol{w}$.

The proofs are tedious but straightforward and we leave them to the interested reader. The construction operates on *finite* sets σ of pairs with each pair of the form

$$\langle S, (v_1, \ldots, v_n) \rangle \in \sigma$$

where $S \subseteq I \times \texttt{Pol}$ is a *set of name-polarity pairs* (with names from the interface I) and $v_1, \ldots, v_n \in (V_{G_\circ} \cup V_{G_\bullet})^*$ is a *finite sequence of vertices* from the two graphs. We call σ a *state* of the construction. We distinguish *consistent* and *live* states based on certain sets of local and global properties, respectively:

Definition 28 (Consistency). *Call a state σ of the construction* consistent *if and only if all pairs $\langle S, (v_1, \ldots, v_n) \rangle \in \sigma$ satisfy the following conditions:*

(i) mutual exclusion: *at most one polarity is assigned to each name in S, i.e. for all $(x, p), (y, q) \in S$, $x = y$ implies $p = q$; equivalently, if $(x, p) \in S$ then $(x, \overline{p}) \notin S$.*

(ii) alternation: *$v_1, \ldots, v_n$ is an alternating path;*
(iii) inclusion: *if $v_1 \in I$ (resp. $v_n \in I$), then there is $p \in \texttt{Pol}$ such that $(v_1, p) \in S$ (resp. $(v_n, p) \in S$);*
(iv) coloring: *if $(v_1, p) \in S$ (resp. $(v_n, p) \in S$), then the alternating path $v_1, \ldots, v_n$ is p-initial (resp. p-final).*

Consider now two sets $S, T \subseteq I \times \texttt{Pol}$ of name-polarity pairs. We say that S, T *compose on name* $x \in \mathcal{N}$ (notation $S \overset{x}{\bowtie} T$) iff there is a polarity $p \in \texttt{Pol}$ such that $(x, p) \in S$, $(x, \overline{p}) \in T$ and $S \setminus \{(x, p)\} = T \setminus \{(x, \overline{p})\}$.

Definition 29 (Liveness). *Call a state σ of the construction* live *iff it is non-empty and, for all $\langle S, \boldsymbol{z} \rangle \in \sigma$ and $(x, p) \in S$, there is $\langle T, \boldsymbol{w} \rangle \in \sigma$ such that $S \overset{x}{\bowtie} T$.*

Now let us define for any state σ the set of *live names of* σ:

$$\texttt{names}(\sigma) = \{x \in \mathcal{N} \mid \langle S, \boldsymbol{z} \rangle \in \sigma, (x, p) \in S\}.$$

Definition 30 (Terminal state). *Call a state σ of the construction* terminal *iff for all $\langle S, \boldsymbol{z} \rangle \in \sigma$, S is empty; equivalently iff $\texttt{names}(\sigma) = \emptyset$.*

Lemma 32. *Let σ be any consistent, live and terminal state: there is a pair $\langle S, \boldsymbol{z} \rangle \in \sigma$ such that $\boldsymbol{z}$ is a complete alternating path.*

Proof. By liveness σ is non-empty, i.e. there is at least one pair $\langle S, \boldsymbol{z} \rangle \in \sigma$. Let $\boldsymbol{z} = z_1, \ldots, z_n$ and assume $z_1 \in I$: by the inclusion condition (consistency) there should be $p \in \texttt{Pol}$ such that $(z_1, p) \in S$, against the hypothesis that σ be terminal, hence $z_1 \notin I$. The same argument shows that $z_n \notin I$. $\square$

Corollary 9. *If there is a consistent, live and terminal state, then there is a complete X-labeled alternating path between $G_\circ$ and $G_\bullet$ through interface I.*

Corollary 10. *If there is a consistent, live and terminal state, then the composite bl-graph $G_\circ \odot_I G_\bullet$ has at least one edge.*

We come now to the key lemma of this proof, showing that every consistent and live state can be transformed into a consistent, live and terminal one in a finite number of steps. Thus we reduce the problem of showing that the composite graph is non-empty to that of constructing a consistent and live state.

Lemma 33 (Reduction lemma). *For any consistent and live state σ such that $\texttt{names}(\sigma)$ is non-empty, there is a consistent and live state τ such that*

$$\texttt{names}(\tau) \subsetneq \texttt{names}(\sigma).$$

Because the interface I is finite by assumption, the set of live names of any state must be finite, therefore we can reach a terminal one by iterating the reduction lemma finitely many times:

Corollary 11. *If there is a consistent and live state, then there is a consistent, live and terminal state.*

Corollary 12. *If there is a consistent and live state, then the composite bl-graph $G_\circ \odot_I G_\bullet$ has at least one edge.*

Proof of the reduction lemma. By assumption there is $x \in \mathtt{names}(\sigma)$. We associate to every pair $\langle S, \boldsymbol{z}\rangle \in \sigma$ a new pair $\langle S', \boldsymbol{z}'\rangle$ constructed as follows, while simultaneously proving that the new pair satisfies all consistency conditions (definition 28):

- if there is no $p \in \mathtt{Pol}$ such that $(x, p) \in S$, then let $S' = S$, $\boldsymbol{z}' = \boldsymbol{z}$; all consistency conditions are clearly preserved;
- otherwise let $\langle T, \boldsymbol{w}\rangle \in \sigma$ be the pair such that $S \overset{x}{\bowtie} T$, whose existence is guaranteed by liveness: we have $p \in \mathtt{Pol}$ such that $(x, p) \in S$ and $(x, \overline{p}) \in T$;
- let $S' = S \setminus \{(x, p)\} = T \setminus \{(x, \overline{p})\}$; since we are just removing one pair, mutual exclusion is preserved;
- let $\boldsymbol{z} = z_1, \ldots, z_n$ and $\boldsymbol{w} = w_1, \ldots, w_n$;
- if both $z_1, z_n \neq x$, then let $\boldsymbol{z}' = \boldsymbol{z}$; the last three consistency conditions are preserved because no endpoint is x;
- if on the other hand $w_1, w_n \neq x$, then let $\boldsymbol{z}' = \boldsymbol{w}$; the last three consistency conditions are preserved because no endpoint is x and $S' = T \setminus \{(x, \overline{p})\}$;
- otherwise there are four mutually exclusive cases:
  - if $z_n = x, w_1 = x$ then let $\boldsymbol{z}' = z_1, \ldots, (z_n = w_1), \ldots, w_n$;
  - if $z_n = x, w_n = x$ then let $\boldsymbol{z}' = z_1, \ldots, (z_n = w_n), \ldots, w_1$;
  - if $z_1 = x, w_1 = x$ then let $\boldsymbol{z}' = z_n, \ldots, (z_1 = w_1), \ldots, w_n$;
  - if $z_1 = x, w_n = x$ then let $\boldsymbol{z}' = z_n, \ldots, (z_1 = w_n), \ldots, w_1$.

  For the inclusion condition, observe that no repetition of vertices is allowed in an alternating path; because the endpoint x has become internal in $\boldsymbol{z}'$, inclusion must still hold for the other endpoints. For alternation: $\boldsymbol{z}'$ is obtained by joining two sequences on x, after possibly reversing them. Reversal preserves alternating paths by lemma 30; the composition is correct (lemma 31) because the two original pairs satisfy the coloring condition, hence the first half is p-final and the second half is $\overline{p}$-initial. Finally, the coloring condition is preserved because the endpoints preserve their original initiality and finality by lemmas 30 and 31.

Clearly the construction is performed in such a way that for any $\langle S, \boldsymbol{z}\rangle \in \sigma$, there is no $p \in \mathtt{Pol}$ such that $(x, p) \in S'$. We define then τ as

$$\tau = \{\langle S', \boldsymbol{z}'\rangle \mid \langle S, \boldsymbol{z}\rangle \in \sigma\}.$$

Since σ is finite, so is τ, hence it is a consistent state. We have obviously

$$\mathtt{names}(\tau) \subseteq \mathtt{names}(\sigma)$$

and from what we said above we known that $x \notin \mathtt{names}(\tau)$, hence the inclusion is strict, as required.

We have now to show that τ is live (definition 29). It must be non-empty because so is σ; let then $\langle S', \boldsymbol{z}' \rangle \in \tau$ and $(y,p) \in S'$: by construction there is some pair $\langle S, \boldsymbol{z} \rangle \in \sigma$ such that $S' = S \setminus \{(x,q)\}$, and by the liveness of σ there is another pair $\langle T, \boldsymbol{w} \rangle \in \sigma$ such that $S \overset{y}{\bowtie} T$. Then there is a pair $\langle T', \boldsymbol{w}' \rangle \in \tau$ such that $T' = T \setminus \{(x,\overline{q})\}$, and we have $S' \overset{y}{\bowtie} T'$: remembering that $y \neq x$,

- $(y,p) \in (S \setminus \{(x,q)\}) = S'$;
- $(y,\overline{p}) \in (T \setminus \{(x,\overline{q})\}) = T'$;
- $S' \setminus \{(y,p)\} = (S \setminus \{(y,p)\}) \setminus \{(x,q)\}$
  $= (T \setminus \{(y,\overline{p})\}) \setminus \{(x,\overline{q})\} = T' \setminus \{(y,\overline{p})\}$. □

As announced above, we complete the proof of lemma 14 by showing that a consistent and live state can be constructed. We start with an auxiliary lemma:

Lemma 34. *For any sharing-free named formula A and function*

$$f : \mathtt{names}(A) \to \mathtt{Pol}$$

there is either $X \in \mathtt{Br}(A)$ such that $fX = \{\circ\}$ or $Y \in \mathtt{Br}(\overline{A})$ such that $fY = \{\bullet\}$.

Proof. By induction on A. If $A = \alpha^x$ then $\mathtt{Br}(A) = \mathtt{Br}(\overline{A}) = \{x\}$, and necessarily either $f\{x\} = \{\circ\}$ or $f\{x\} = \{\bullet\}$.

If $A = B \wedge C$ (with $\overline{A} = \overline{B} \vee \overline{C}$) then $\mathtt{Br}(A) = \mathtt{Br}(B) \cup \mathtt{Br}(C))$. By induction hypothesis there is either $X' \in \mathtt{Br}(B)$ such that $fX' = \{\circ\}$ or $Y' \in \mathtt{Br}(\overline{B})$ such that $fY' = \{\bullet\}$. In the first case, let $X = X'$ and we're done. Otherwise, there is again by induction hypothesis either $X'' \in \mathtt{Br}(C)$ s.t. $fX'' = \{\circ\}$, or $Y'' \in \mathtt{Br}(\overline{C})$ s.t. $fY'' = \{\bullet\}$. If there is such a X'', let $X = X''$; otherwise, let $Y = Y' \cup Y'' \in \mathtt{Br}(\overline{B} \vee \overline{C})$: we have $fY = fY' \cup fY'' = \{\bullet\}$.

If $A = B \vee C$, apply the same argument, swapping polarities and A with $\overline{A}$. □

Proof of lemma 14. We fix $G_\circ = (\!|P|\!)$, $G_\bullet = (\!|Q|\!)$ and $I = \mathtt{names}(A)$. I is obviously finite, and because the context of the conclusions of P, Q is atomic by hypothesis, the condition about branches is satisfied too (the unique branch name X is the unique one in $\mathtt{Br}(\Gamma)$, i.e. $\mathtt{names}(\Gamma)$).

We construct a state σ as follows:

$$\sigma = \{\langle f, (x,y) \rangle \mid f \in \mathtt{Pol}^I, p \in \mathtt{Pol}, xy \prec_{G_p} X \cup Y, fY = \{p\}\}.$$

Let us unpack the construction first, then we shall prove that the state is consistent and live. $\mathtt{Pol}^I$ is the set of all functions from $I = \mathtt{names}(A)$ to $\mathtt{Pol}$ seen as sets of pairs, i.e. $\mathtt{Pol}^I \subseteq I \times \mathtt{Pol}$; we pair each $f \in \mathtt{Pol}^I$ with any finite sequence x, y of vertices such that, for at least one polarity $p \in \mathtt{Pol}$, xy is an edge in G_p, and moreover one of its branch labels satisfies the following conditions:

- it is equal to $X \cup Y$ for some Y (where X is the unique branch name of Γ described above);
- it is such that $fY = \{p\}$.

σ is clearly finite, hence a state. For consistency (definition 28), let $\langle f, (x,y) \rangle \in \sigma$:

- *mutual exclusion:* this condition can be read as asking that f be the graph of a function, hence it is obviously satisfied;
- *alternation:* xy is an edge in either $G_\circ$ or $G_\bullet$ by construction, hence an alternating path by lemma 29;
- *inclusion:* because f is total, if $x \in I$ then $(x, fx) \in f$, and similarly for y;
- *coloring:* let $(x, p) \in f$; by construction there is $q \in \texttt{Pol}$, $Y \in \mathcal{N}$ such that $xy \prec_{G_q} X \cup Y$ with $fY = \{q\}$. By lemma 8, every branch name of G_q is the union of the unique branch name X of Γ with some branch name of either A (if $q = \circ$) or $\overline{A}$ (if $q = \bullet$). If $q = \circ$, then, $Y \in \texttt{Br}(A)$, otherwise $Y \in \texttt{Br}(\overline{A})$. By definition 11 $x \in X \cup Y$, and because $x \in I$ we must have $x \in Y$. We know that $fY = \{q\}$, hence in particular $p = f(x) = q$. The path x, y is obviously q-initial, hence p-initial. Analogous reasoning shows that if $(y, p') \in f$, then the path is p'-final.

For liveness (definition 29) we argue first that σ is non-empty: by lemma 34 above, there is for each $f \in \texttt{Pol}^I$ at least one branch $Y \in \texttt{Br}(A)$ with $fY = \{\circ\}$ or $Y \in \texttt{Br}(\overline{A})$ with $fY = \{\bullet\}$. Let $fY = \{p\}$; by lemma 17, $(X \cup Y) \in \texttt{Br}(G_p)$, i.e. there is $e \prec_{G_p} X \cup Y$. We have thus proven not just that σ is non-empty, but also that for each $f \in \texttt{Pol}^I$ there is at least one pair $\langle f, \boldsymbol{z} \rangle \in \sigma$.

Now let $\langle f, \boldsymbol{z} \rangle \in \sigma$, $(x, p) \in f$; by inverting the polarity assignment of x in f we obtain $f' = (f \setminus \{(x, p)\}) \cup \{(x, \overline{p})\} \in \texttt{Pol}^I$. By the reasoning above there is $\langle f', \boldsymbol{w} \rangle \in \sigma$, and it is immediate by construction that $f \overset{x}{\bowtie} f'$. □

H Totality lemmas and correctness algorithm for BLG

Proof of lemma 17. By structural induction on P. If P ends with an axiom rule application, the conclusion is equivalent to corollary 8; if P ends with a superposition rule application, it follows immediately from the induction hypothesis; if P ends with a logical rule, we conclude from the induction hypothesis and proposition 3, points (ii) and (iii). The details are left to the reader. □

Proof of lemma 18. We have $\texttt{names}(\Gamma, A, B) = \texttt{names}(\Gamma, A \vee B)$; $\texttt{Br}(\Gamma, A, B) = \texttt{Br}(\Gamma, A \vee B)$ by proposition 3; and finally $(\Gamma, A, B)[x] = (\Gamma, A \vee B)[x]$ for all $x \in \texttt{names}(\Gamma, A, B)$. □

Proof of lemma 19. We have two prove two facts: *(i)* that G is effectively equal to the union of the two restrictions, and *(ii)* that the two restrictions are total w.r.t. the restricted conclusions. For fact (i), we have obviously

$$\texttt{names}(\Gamma, A \wedge B) = \texttt{names}(\Gamma, A) \cup \texttt{names}(\Gamma, B).$$

Then by construction $V_{G\restriction_{\Gamma,A}} = \texttt{names}(\Gamma, A)$ and $V_{G\restriction_{\Gamma,B}} = \texttt{names}(\Gamma, B)$, hence $V_G = V_{G\restriction_{\Gamma,A}} \cup V_{G\restriction_{\Gamma,B}}$. Now let $e \prec_G X$. By totality $X \in \texttt{Br}(\Gamma, A \wedge B)$; then by proposition 3 either $X \in \texttt{Br}(\Gamma, A)$ or $X \in \texttt{Br}(\Gamma, B)$: in the first case we have $X \subseteq \texttt{names}(\Gamma, A)$ hence $e \prec_{G\restriction_{\Gamma,A}} X$; similarly in the second case we have $e \prec_{G\restriction_{\Gamma,B}} X$. The reverse inclusion is obvious.

For fact (ii), we have already argued that the two restriction have the appropriate vertex sets (definition 17, point (i)); that all edges link dual atoms (definition 17, point (iii)) follows from the fact that the edges come from G, which satisfies the same condition, and for all $x \in \mathtt{names}(\Gamma, A)$ (resp. $\mathtt{names}(\Gamma, B)$) we have $(\Gamma, A \wedge B)[x] = (\Gamma, A)[x]$ (resp. $(\Gamma, B)[x]$). Finally we have to show that $\mathtt{Br}(G{\restriction_{\Gamma,A}}) = \mathtt{Br}(\Gamma, A)$ and $\mathtt{Br}(G{\restriction_{\Gamma,B}}) = \mathtt{Br}(\Gamma, B)$ (definition 17, point (ii)). The forward inclusions are proved already in lemma 28. For the reverse, let $X \in \mathtt{Br}(\Gamma, A)$ (resp. $\mathtt{Br}(\Gamma, B)$); by totality there is $e \prec_G X$, and because $X \subseteq \mathtt{names}(\Gamma, A)$ (resp. $\mathtt{names}(\Gamma, B)$) we have $e \prec_{G\restriction_{\Gamma,A}} X$ (resp. $e \prec_{G\restriction_{\Gamma,B}} X$). □

H.1 Correctness algorithm for BLG

Proof of proposition 5. We work under the reasonable assumption that checking name equality requires constant time. Let us start by recalling the definition of the size of $\mathbf{G}$ (definition 19):

$$\mathtt{size}(\mathbf{G}) = \mathtt{size}(\vdash \Gamma) + |V_G| + \sum_{e \prec_G X} |X|.$$

The size of $\vdash \Gamma$ (definition 24, appendix B) is by proposition 9 the sum of the number of atom occurrences in Γ (notation $\#\mathtt{at}(\vdash \Gamma)$) with the number of logical symbols, also called the *degree of* Γ (notation $\mathtt{deg}(\Gamma)$). Because Γ is sharing-free by hypothesis, the number of atom occurrences coincides with the number of names, i.e. $\#\mathtt{at}(\vdash \Gamma) = |\mathtt{names}(\Gamma)|$.

Observe also that the sum at the end of the expression is taken not over the set of branches of G, but over all edge-branch pairs, i.e. a branch may be counted more than once if it has multiple edges. Because $e \prec_G X$ implies $e \subseteq X$, i.e. no branch in a bl-graph is empty, the sum provides an upper bound to the number of edge-branch pairs as well as to that of edges and branches, i.e. we have

$$|E_G|, |\mathtt{Br}(G)| \leq |{\prec_G}| \leq \mathtt{size}(\mathbf{G}).$$

We have to check three conditions separately (definition 17):

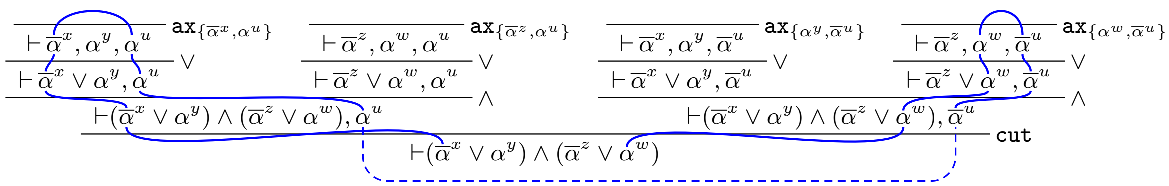
(i) $V_G = \mathtt{names}(\Gamma)$: checking name set equality is worst-case polynomial in their cardinalities, and constructing the set $\mathtt{names}(\Gamma)$ from Γ requires polynomial time in the size of $\vdash \Gamma$;

(ii) $\mathtt{Br}(G) = \mathtt{Br}(\Gamma)$: this is the most complex problem. Generating just one element in $\mathtt{Br}(\Gamma)$ amounts to persistently expanding one branch of a $\mathtt{GS4}^{\mathcal{N}}$ derivation of Γ until an atomic sequent is reached; the length of such a branch is known to be bounded by the complexity degree of $\vdash \Gamma$ [30], hence the cost of generating one branch name is polynomially bounded in the size of $\vdash \Gamma$. However, the total number of branches is in the worst case exponential in the complexity degree of $\vdash \Gamma$: thus we cannot take the naive approach of constructing the whole set $\mathtt{Br}(\Gamma)$, then testing for equality. We describe an informal algorithm: the idea is to generate the branch names of Γ incrementally, match them with some element from the set $\mathtt{Br}(G)$ and erase that element. For every matching attempt, either the algorithm fails or the number of branches still to be matched decreases: in this way the number of generated branch names is always bounded by the cardinality of $\mathtt{Br}(G)$. Let S be the branch set to test (initially $S = \mathtt{Br}(G)$), Δ the sequent to test against (initially $\Delta = \Gamma$); the algorithm has three phases:

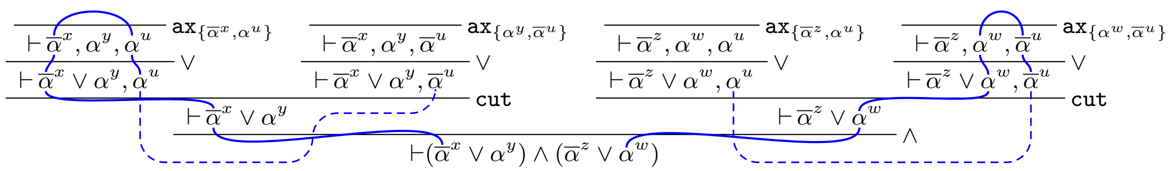
- *problem reduction phase:* if the sequent Δ to test against contains a disjunction, i.e. is of the form $\Delta', A \vee B$, then replace it with Δ', A, B. If it contains no disjunction but at least a conjunction, i.e. is of the form $\Delta', A \wedge B$, then replace it with Δ', A and append Δ', B to a list of sequents to test against later. By proposition 3, the set of branches to test against has not changed;
- *matching phase:* if the sequent Δ to test against is atomic, then it has a unique branch $X = \mathtt{names}(\Delta)$. We search for that branch in S: if not found, then $X \in \mathtt{Br}(\Gamma)$ but $X \notin \mathtt{Br}(G)$, and we stop; if found, we erase X from S and move on to the backtracking phase;
- *backtracking phase:* if both S and the list of delayed sequents are empty, then we're done. If S is empty but the list is not, then some branches are missing from $\mathtt{Br}(G)$ and we stop. If S has some elements but the list is empty, then there are excess branches in $\mathtt{Br}(G)$ and we stop. If both are non-empty, we pick the first element of the list as the new Δ, erase it from the list and move back to the reduction phase.

It is clear that the number of steps in the reduction phase is bounded by the complexity degree of the active sequent Δ, which always decreases: when it reaches zero, we move to the matching phase. The matching phase either fails or decreases the size of the set B; once the set B becomes empty, the algorithm stops: thus the size of B bounds the number of future reduction phases. Observe now that a reduction step requires polynomial time in the size of the active sequent; a matching step requires polynomial time in the cardinality of the generated branch and the sum of the cardinalities of all elements of B. Backtracking steps require constant time. Every measure mentioned above is itself bounded by $\mathtt{size}(\mathbf{G})$. We have then a polynomial bound on the execution time of the algorithm with parameter $(\mathtt{size}(\mathbf{G}))^3$.

(iii) for all $xy \in E_G$, $\Gamma[x] = \overline{\Gamma[y]}$: we observed at the beginning of the proof that $\mathtt{size}(\mathbf{G})$ bounds $|E_G|$; finding the atom associated to a given name in Γ is polynomial in the size of $\vdash \Gamma$. □



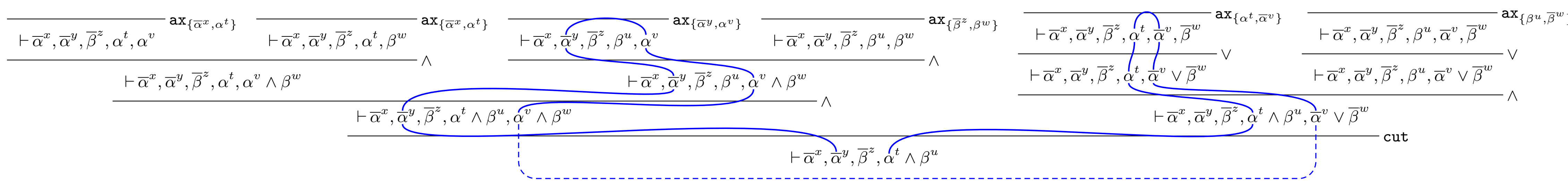
(a) The initial derivation, where the conjunction in the conclusion is not introduced by the last rule. Assume names $x, y, z, w, u \in \mathcal{N}$ are pairwise distinct.



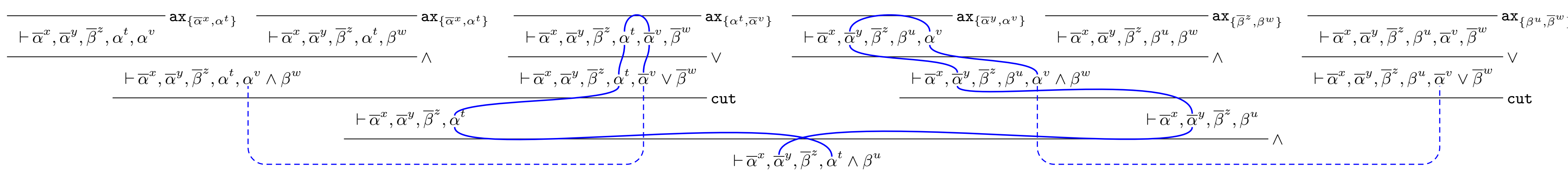
(b) The transformed derivation, after isolating the conjuction. The two halves of the original alternating path are now disconnected.

(c) The axiom graph of the original derivation (left) and that of the transformed one (right).

Fig. 4: A derivation with cuts whose axiom graph decreases when isolating the conjunction in its conclusion.



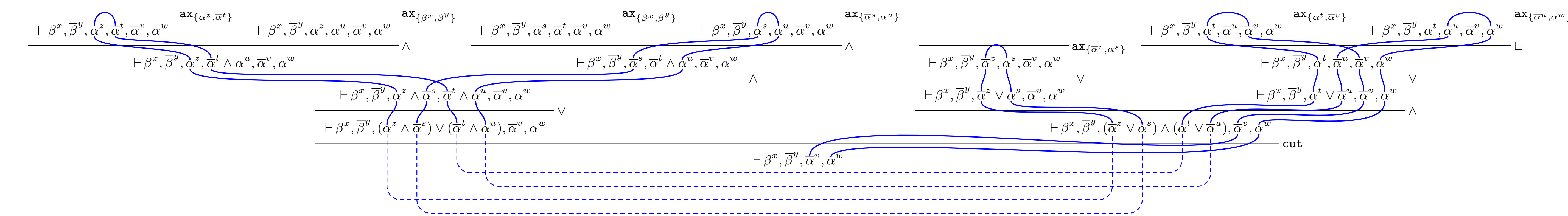
(a) The initial derivation, where the conjunction in the conclusion is not introduced by the last rule. Assume names $x, y, z, t, u, v, w \in \mathcal{N}$ are pairwise distinct.



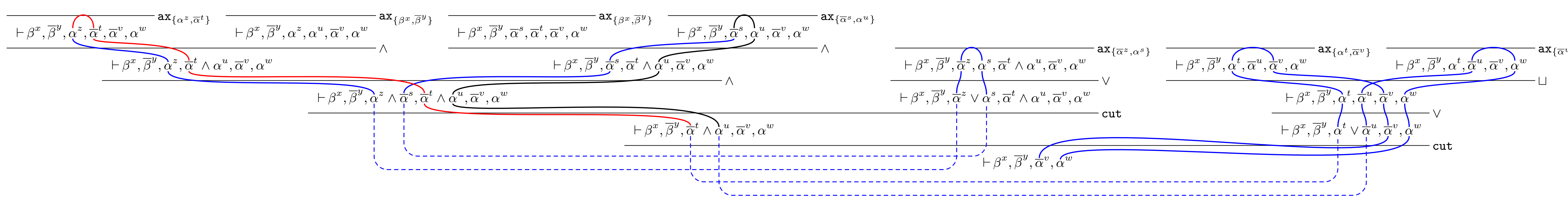
(b) The transformed derivation, after isolating the conjuction. The two halves of the original alternating path are now disconnected.

(c) The axiom graph of the original derivation (left) and that of the transformed one (right).

Fig. 5: A derivation with cuts whose axiom graph decreases when isolating the conjunction in its conclusion. The lost path does not visibly cross a conjunction.



(a) The initial derivation, with a unique cut-rule. Assume names $x, y, z, s, t, u, v, w \in \mathcal{N}$ are pairwise distinct. There is an alternating path connecting $\overline{\alpha}^v$ with α^w. Because the conclusion is atomic, there is only one branch name up to names in the interface, hence all edges are compatible.



(b) A logical cut-reduction step has been applied (see section 8). The conjunction $\overline{\alpha}^t \wedge \alpha^u$ is now outside the interface of the upper cut: the alternating path that connects $\overline{\alpha}^t$ with α^u must be omitted when computing the interpretation, as it uses edges from incompatible branches (highlighted in red and black). As a result, the edge between $\overline{\alpha}^v$ and α^w is lost.

(c) The branch-labeled axiom graph of the original derivation (left) and that of the reduced one (right).

Fig. 6: A derivation whose branch-labeled axiom graph decreases when a logical cut-reduction step is applied.