ON THE SEMANTICS OF PROOFS IN CLASSICAL SEQUENT CALCULUS

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GENTZEN'S CALCULUS LK (PROPOSITIONAL FRAGMENT)

$$F, G ::= \alpha \mid \neg F \mid F \to G \mid F \vee G \mid F \wedge G \qquad (\alpha \in \mathcal{A})$$

Identity group:

$$\frac{}{F \vdash F}\,\texttt{ax} \qquad \frac{\Gamma \vdash \Delta, F \quad F, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\texttt{cut}$$

Structural group:

$$\frac{\Gamma, F, G, \Gamma' \vdash \Delta}{\Gamma, G, F, \Gamma' \vdash \Delta}\,\texttt{xch}\vdash \qquad \frac{\Gamma \vdash \Delta}{F, \Gamma \vdash \Delta}\,\texttt{wk}\vdash \qquad \frac{F, F, \Gamma \vdash \Delta}{F, \Gamma \vdash \Delta}\,\texttt{ctr}\vdash$$

$$\frac{\Gamma \vdash \Delta, F, G\Delta'}{\Gamma \vdash \Delta, G, F, \Delta'}\vdash\texttt{xch} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, F}\vdash\texttt{wk} \qquad \frac{\Gamma \vdash \Delta, F, F}{\Gamma \vdash \Delta, F, F}\vdash\texttt{ctr}$$

Additive group (context sharing):

$$\frac{F, \Gamma \vdash \Delta}{F \wedge G, \Gamma \vdash \Delta}\wedge_{\texttt{l}}\vdash \qquad \frac{\Gamma \vdash \Delta, F}{\Gamma \vdash \Delta, F \vee G}\vdash\vee_{\texttt{l}}$$

$$\frac{G, \Gamma \vdash \Delta}{F \wedge G, \Gamma \vdash \Delta}\wedge_{\texttt{r}}\vdash \qquad \frac{\Gamma \vdash \Delta, G}{\Gamma \vdash \Delta, F \vee G}\vdash\vee_{\texttt{r}}$$

$$\frac{\Gamma \vdash \Delta, F \quad \Gamma \vdash \Delta, G}{\Gamma \vdash \Delta, F \wedge G}\vdash\wedge \qquad \frac{F, \Gamma \vdash \Delta \quad G, \Gamma \vdash \Delta}{F \vee G, \Gamma \vdash \Delta}\vee\vdash$$

Multiplicative group (context splitting):

$$\frac{\Gamma \vdash \Delta, F}{\neg F, \Gamma \vdash \Delta}\neg\vdash \qquad \frac{\Gamma \vdash \Delta, F \quad G, \Gamma' \vdash \Delta'}{F \to G, \Gamma, \Gamma' \vdash \Delta, \Delta'}\to\vdash$$

$$\frac{F, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg F}\vdash\neg \qquad \frac{F, \Gamma \vdash \Delta, G}{\Gamma \vdash \Delta, F \to G}\vdash\to$$

THE CALCULUS LK (PROPOSITIONAL FRAGMENT, ONE SIDED)

$$F, G ::= \alpha \mid \overline{\alpha} \mid F \vee G \mid F \wedge G \qquad (\alpha \in \mathcal{A})$$

$$\overline{(\alpha)} = \overline{\alpha} \qquad \overline{(\overline{\alpha})} = \alpha \qquad \overline{F \vee G} = \overline{F} \wedge \overline{G} \qquad \overline{F \wedge G} = \overline{F} \vee \overline{G}$$

Identity group:

$$\frac{}{\vdash \overline{A}, A}\,\texttt{ax} \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, \overline{A}}{\vdash \Gamma, \Delta}\,\texttt{cut}$$

Structural group:

$$\frac{\vdash \Gamma}{\vdash \Gamma, A}\,\texttt{wk} \qquad \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\,\texttt{ctr}$$

Logical group:

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}\,\vee \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \wedge B}\,\wedge$$

CUT-ELIMINATION THEOREMS

Coarse form. *The cut-rule is admissible.*

Refined form. *For every derivation* $P \vdash \Gamma$ *there is a cut-free derivation* $Q \vdash \Gamma$ *such that* $P \longrightarrow^* Q$.

Denotational semantics:

Define functions $[\![-]\!] : \mathtt{LK} \to X$ mapping derivations to *denotations*, such that

$$P \longrightarrow Q \implies [\![P]\!] = [\![Q]\!]$$

CUT-ELIMINATION THEOREMS

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Interesting invariants:

Coarsest – correctness and conclusions.

Finest – normal form (if unique).

CUT-ELIMINATION THEOREMS

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CUT-ELIMINATION THEOREMS

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Interesting invariants:

Coarsest – correctness and conclusions.

Finest – normal form (if unique).

Intermediate ones?

CUT-ELIMINATION — KEY CASES

$$\dfrac{\dfrac{\overset{\vdots\; P}{\vdash \Gamma, \overline{F}, \overline{G}}}{\vdash \Gamma, \overline{F} \vee \overline{G}}\,\vee \qquad \dfrac{\overset{\vdots\; Q}{\vdash \Delta_1, F} \quad \overset{\vdots\; R}{\vdash \Delta_2, G}}{\vdash \Delta_1, \Delta_2, F \wedge G}\,\wedge}{\vdash \Gamma, \Delta_1, \Delta_2}\,\texttt{cut} \quad \longrightarrow \quad \dfrac{\dfrac{\overset{\vdots\; P}{\vdash \Gamma, \overline{F}, \overline{G}} \quad \overset{\vdots\; Q}{\vdash \Delta_1, F}}{\vdash \Gamma, \Delta_1, \overline{G}}\,\texttt{cut} \qquad \overset{\vdots\; R}{\vdash \Delta_2, G}}{\vdash \Gamma, \Delta_1, \Delta_2}\,\texttt{cut}$$

$$\dfrac{\dfrac{\overset{\vdots\; P}{\vdash \Gamma, \overline{F}, \overline{G}}}{\vdash \Gamma, \overline{F} \vee \overline{G}}\,\vee \qquad \dfrac{\overset{\vdots\; Q}{\vdash \Delta_1, F} \quad \overset{\vdots\; R}{\vdash \Delta_2, G}}{\vdash \Delta_1, \Delta_2, F \wedge G}\,\wedge}{\vdash \Gamma, \Delta_1, \Delta_2}\,\texttt{cut} \quad \longrightarrow \quad \dfrac{\dfrac{\overset{\vdots\; P}{\vdash \Gamma, \overline{F}, \overline{G}} \quad \overset{\vdots\; R}{\vdash \Delta_2, G}}{\vdash \Gamma, \overline{F}, \Delta_2}\,\texttt{cut} \qquad \overset{\vdots\; Q}{\vdash \Delta_1, F}}{\vdash \Gamma, \Delta_1, \Delta_2}\,\texttt{cut}$$

CUT-ELIMINATION — STRUCTURAL CASES

$$\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma}}{\vdash \Gamma, F}\,\mathtt{wk} \qquad \overset{\vdots\, Q}{\vdash \Delta, \overline{F}}}{\vdash \Gamma, \Delta}\,\mathtt{cut} \quad\longrightarrow\quad \dfrac{\overset{\vdots\, P}{\vdash \Gamma}}{\vdash \Gamma, \Delta}\,\mathtt{wk}$$

$$\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma, F, F}}{\vdash \Gamma, F}\,\mathtt{ctr} \qquad \overset{\vdots\, Q}{\vdash \Delta, \overline{F}}}{\vdash \Gamma, \Delta}\,\mathtt{cut} \quad\longrightarrow\quad \dfrac{\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma, F, F} \quad \overset{\vdots\, Q}{\vdash \Delta, \overline{F}}}{\vdash \Gamma, \Delta}\,\mathtt{cut} \qquad \overset{\vdots\, Q}{\vdash \Delta, \overline{F}}}{\vdash \Gamma, \Delta, \Delta}\,\mathtt{cut}}{\vdash \Gamma, \Delta}\,\mathtt{ctr}$$

CUT-ELIMINATION — COMMUTATIVE CASES (EXAMPLES)

$$\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma, F, G, H}}{\vdash \Gamma, F \vee G, H}\,\vee \qquad \overset{\vdots\, Q}{\vdash \Delta, \overline{H}}}{\vdash \Gamma, F \vee G, \Delta}\,\texttt{cut} \quad\longrightarrow\quad \dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma, F, G, H} \quad \overset{\vdots\, Q}{\vdash \Delta, \overline{H}}}{\vdash \Gamma, F, G, \Delta}\,\texttt{cut}}{\vdash \Gamma, F \vee G, \Delta}\,\vee$$

$$\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma_1, F, H} \quad \overset{\vdots\, Q}{\vdash \Gamma_2, G}}{\vdash \Gamma_1, \Gamma_2, F \wedge G, H}\,\wedge \qquad \overset{\vdots\, R}{\vdash \Delta, \overline{H}}}{\vdash \Gamma_1, \Gamma_2, F \wedge G, \Delta}\,\texttt{cut} \quad\longrightarrow\quad \dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma_1, F, H} \quad \overset{\vdots\, R}{\vdash \Delta, \overline{H}}}{\vdash \Gamma_1, F, \Delta}\,\texttt{cut} \qquad \overset{\vdots\, Q}{\vdash \Gamma_2, G}}{\vdash \Gamma_1, \Gamma_2, F \wedge G, \Delta}\,\wedge$$

PATHOLOGICAL CRITICAL PAIRS

$$\dfrac{\dfrac{\overset{\vdots\;P}{\vdash \Gamma,\overline{F},\overline{G}} \quad \overset{\vdots\;Q}{\vdash \Delta_1,F}}{\vdash \Gamma,\Delta_1,\overline{G}}\,\texttt{cut} \qquad \overset{\vdots\;R}{\vdash \Delta_2,G}}{\vdash \Gamma,\Delta_1,\Delta_2}\,\texttt{cut} \quad\longleftarrow\quad \dfrac{\dfrac{\overset{\vdots\;P}{\vdash \Gamma,\overline{F},\overline{G}}}{\vdash \Gamma,\overline{F}\vee\overline{G}}\,\vee \qquad \dfrac{\overset{\vdots\;Q}{\vdash \Delta_1,F} \quad \overset{\vdots\;R}{\vdash \Delta_2,G}}{\vdash \Delta_1,\Delta_2,F\wedge G}\,\wedge}{\vdash \Gamma,\Delta_1,\Delta_2}\,\texttt{cut} \quad\longrightarrow\quad \dfrac{\dfrac{\overset{\vdots\;P}{\vdash \Gamma,\overline{F},\overline{G}} \quad \overset{\vdots\;R}{\vdash \Delta_2,G}}{\vdash \Gamma,\overline{F},\Delta_2}\,\texttt{cut} \qquad \overset{\vdots\;Q}{\vdash \Delta_1,F}}{\vdash \Gamma,\Delta_1,\Delta_2}\,\texttt{cut}$$

$$\dfrac{\overset{\vdots\;P}{\vdash \Gamma}}{\vdash \Gamma,\Delta}\,\texttt{wk} \quad\longleftarrow\quad \dfrac{\dfrac{\overset{\vdots\;P}{\vdash \Gamma}}{\vdash \Gamma,F}\,\texttt{wk} \qquad \dfrac{\overset{\vdots\;Q}{\vdash \Delta}}{\vdash \Delta,\overline{F}}\,\texttt{wk}}{\vdash \Gamma,\Delta}\,\texttt{cut} \quad\longrightarrow\quad \dfrac{\overset{\vdots\;Q}{\vdash \Delta}}{\vdash \Gamma,\Delta}\,\texttt{wk}$$

$$\dfrac{\dfrac{\dfrac{\overset{\vdots\;P}{\vdash \Gamma,F,F} \quad \dfrac{\overset{\vdots\;Q}{\vdash \Delta,\overline{F},\overline{F}}}{\vdash \Delta,\overline{F}}\,\texttt{ctr}}{\vdash \Gamma,\Delta}\,\texttt{cut} \qquad \dfrac{\overset{\vdots\;Q}{\vdash \Delta,\overline{F},\overline{F}}}{\vdash \Delta,\overline{F}}\,\texttt{ctr}}{\vdash \Gamma,\Delta,\Delta}\,\texttt{cut}}{\vdash \Gamma,\Delta}\,\texttt{ctr} \quad\longleftarrow\quad \dfrac{\dfrac{\overset{\vdots\;P}{\vdash \Gamma,F,F}}{\vdash \Gamma,F}\,\texttt{ctr} \qquad \dfrac{\overset{\vdots\;Q}{\vdash \Delta,\overline{F},\overline{F}}}{\vdash \Delta,\overline{F}}\,\texttt{ctr}}{\vdash \Gamma,\Delta}\,\texttt{cut} \quad\longrightarrow\quad \dfrac{\dfrac{\dfrac{\overset{\vdots\;P}{\vdash \Gamma,F,F}}{\vdash \Gamma,F}\,\texttt{ctr} \qquad \dfrac{\dfrac{\overset{\vdots\;P}{\vdash \Gamma,F,F}}{\vdash \Gamma,F}\,\texttt{ctr} \quad \overset{\vdots\;Q}{\vdash \Delta,\overline{F},\overline{F}}}{\vdash \Gamma,\Delta}\,\texttt{cut}}{\vdash \Gamma,\Gamma}\,\texttt{cut}}{\vdash \Gamma,\Delta}\,\texttt{ctr}$$

PATHOLOGICAL CRITICAL PAIRS

$$\dfrac{\dfrac{\begin{matrix}\vdots\, P\\ \vdash \Gamma,\overline{F},\overline{G}\end{matrix} \quad \begin{matrix}\vdots\, Q\\ \vdash \Delta_1,F\end{matrix}}{\vdash \Gamma,\Delta_1,\overline{G}}\,\mathtt{cut} \quad \begin{matrix}\vdots\, R\\ \vdash \Delta_2,G\end{matrix}}{\vdash \Gamma,\Delta_1,\Delta_2}\,\mathtt{cut} \;\longleftarrow\; \dfrac{\dfrac{\begin{matrix}\vdots\, P\\ \vdash \Gamma,\overline{F},\overline{G}\end{matrix}}{\vdash \Gamma,\overline{F}\vee\overline{G}}\,\vee \quad \dfrac{\begin{matrix}\vdots\, Q\\ \vdash \Delta_1,F\end{matrix} \quad \begin{matrix}\vdots\, R\\ \vdash \Delta_2,G\end{matrix}}{\vdash \Delta_1,\Delta_2,F\wedge G}\,\wedge}{\vdash \Gamma,\Delta_1,\Delta_2}\,\mathtt{cut} \;\longrightarrow\; \dfrac{\dfrac{\begin{matrix}\vdots\, P\\ \vdash \Gamma,\overline{F},\overline{G}\end{matrix} \quad \begin{matrix}\vdots\, R\\ \vdash \Delta_2,G\end{matrix}}{\vdash \Gamma,\overline{F},\Delta_2}\,\mathtt{cut} \quad \begin{matrix}\vdots\, Q\\ \vdash \Delta_1,F\end{matrix}}{\vdash \Gamma,\Delta_1,\Delta_2}\,\mathtt{cut}$$

$$\dfrac{\begin{matrix}\vdots\, P\\ \vdash \Gamma\end{matrix}}{\vdash \Gamma,\Delta}\,\mathtt{wk} \;\longleftarrow\; \dfrac{\dfrac{\begin{matrix}\vdots\, P\\ \vdash \Gamma\end{matrix}}{\vdash \Gamma,F}\,\mathtt{wk} \quad \dfrac{\begin{matrix}\vdots\, Q\\ \vdash \Delta\end{matrix}}{\vdash \Delta,\overline{F}}\,\mathtt{wk}}{\vdash \Gamma,\Delta}\,\mathtt{cut} \;\longrightarrow\; \dfrac{\begin{matrix}\vdots\, Q\\ \vdash \Delta\end{matrix}}{\vdash \Gamma,\Delta}\,\mathtt{wk}$$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\, P\\ \vdash \Gamma,F,F\end{matrix} \quad \dfrac{\begin{matrix}\vdots\, Q\\ \vdash \Delta,\overline{F},\overline{F}\end{matrix}}{\vdash \Delta,\overline{F}}\,\mathtt{ctr}}{\vdash \Gamma,\Delta}\,\mathtt{cut} \quad \dfrac{\begin{matrix}\vdots\, Q\\ \vdash \Delta,\overline{F},\overline{F}\end{matrix}}{\vdash \Delta,\overline{F}}\,\mathtt{ctr}}{\vdash \Gamma,\Delta,\Delta}\,\mathtt{cut}}{\vdash \Gamma,\Delta}\,\mathtt{ctr} \;\longleftarrow\; \dfrac{\dfrac{\begin{matrix}\vdots\, P\\ \vdash \Gamma,F,F\end{matrix}}{\vdash \Gamma,F}\,\mathtt{ctr} \quad \dfrac{\begin{matrix}\vdots\, Q\\ \vdash \Delta,\overline{F},\overline{F}\end{matrix}}{\vdash \Delta,\overline{F}}\,\mathtt{ctr}}{\vdash \Gamma,\Delta}\,\mathtt{cut} \;\longrightarrow\; \dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\, P\\ \vdash \Gamma,F,F\end{matrix}}{\vdash \Gamma,F}\,\mathtt{ctr} \quad \dfrac{\dfrac{\begin{matrix}\vdots\, P\\ \vdash \Gamma,F,F\end{matrix}}{\vdash \Gamma,F}\,\mathtt{ctr} \quad \begin{matrix}\vdots\, Q\\ \vdash \Delta,\overline{F},\overline{F}\end{matrix}}{\vdash \Gamma,\Delta}\,\mathtt{cut}}{\vdash \Gamma,\Gamma}\,\mathtt{cut}}{\vdash \Gamma,\Delta}\,\mathtt{ctr}$$

LAFONT'S EXAMPLE & CO. (PROOFS AND TYPES, 1989)

$$\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma}}{\vdash \Gamma, \Gamma}\,\mathtt{wk}}{\vdash \Gamma}\,\mathtt{ctr} \;=\; P' \quad\longleftarrow\quad \dfrac{\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma}}{\vdash \Gamma, F}\,\mathtt{wk} \qquad \dfrac{\overset{\vdots\, Q}{\vdash \Gamma}}{\vdash \Gamma, \overline{F}}\,\mathtt{wk}}{\vdash \Gamma, \Gamma}\,\mathtt{cut}}{\vdash \Gamma}\,\mathtt{ctr} \quad\longrightarrow\quad Q' \;=\; \dfrac{\dfrac{\overset{\vdots\, Q}{\vdash \Gamma}}{\vdash \Gamma, \Gamma}\,\mathtt{wk}}{\vdash \Gamma}\,\mathtt{ctr}$$

Assumption 1. *There is a denotational interpretation* $[\![-]\!] : \mathtt{LK} \to X$ *mapping LK derivations to denotations in some space X.*

Assumption 2. *The interpretation is such that* $[\![P']\!] = [\![P]\!]$ *and* $[\![Q']\!] = [\![Q]\!]$.

Fact. $[\![P']\!] = [\![Q']\!]$, *hence* $[\![P]\!] = [\![Q]\!]$.

LAFONT'S EXAMPLE & CO. (PROOFS AND TYPES, 1989)

$$\dfrac{\dfrac{\begin{matrix}\vdots\, P\\ \vdash F\end{matrix}}{\vdash F, G}\,\mathtt{wk}}{\vdash F \vee G}\,\vee \;= P' \quad\longleftarrow\quad \dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\, P\\ \vdash F\end{matrix}}{\vdash F, H}\,\mathtt{wk} \qquad \dfrac{\begin{matrix}\vdots\, Q\\ \vdash G\end{matrix}}{\vdash G, \overline{H}}\,\mathtt{wk}}{\vdash F, G}\,\mathtt{cut}}{\vdash F \vee G}\,\vee \quad\longrightarrow\quad Q' = \dfrac{\dfrac{\begin{matrix}\vdots\, Q\\ \vdash G\end{matrix}}{\vdash F, G}\,\mathtt{wk}}{\vdash F \vee G}\,\vee$$

Assumption 1. *There is a denotational interpretation* $[\![-]\!] : \mathtt{LK} \to X$ *mapping LK derivations to denotations in some space X.*

Fact. $[\![P']\!] = [\![Q']\!]$.

LAFONT'S EXAMPLE & CO. (PROOFS AND TYPES, 1989)

$$
\dfrac{\begin{array}{c}\vdots\, P\\ \vdash \Gamma\end{array} \quad \begin{array}{c}\vdots\, Q\\ \vdash \Delta\end{array}}{\vdash \Gamma, \Delta}\,\texttt{mix}
\quad \overset{?}{\longleftarrow} \quad
\dfrac{\dfrac{\begin{array}{c}\vdots\, P\\ \vdash \Gamma\end{array}}{\vdash \Gamma, F}\,\texttt{wk} \quad \dfrac{\begin{array}{c}\vdots\, Q\\ \vdash \Delta\end{array}}{\vdash \Delta, \overline{F}}\,\texttt{wk}}{\vdash \Gamma, \Delta}\,\texttt{cut}
\quad \overset{?}{\longrightarrow} \quad
\dfrac{\dfrac{\begin{array}{c}\vdots\, P\\ \vdash \Gamma\end{array}}{\vdash \Gamma, \Delta}\,\texttt{wk} \quad \dfrac{\begin{array}{c}\vdots\, Q\\ \vdash \Delta\end{array}}{\vdash \Gamma, \Delta}\,\texttt{wk}}{\vdash \Gamma, \Delta}\,\oplus
$$

PATHOLOGICAL CRITICAL PAIRS

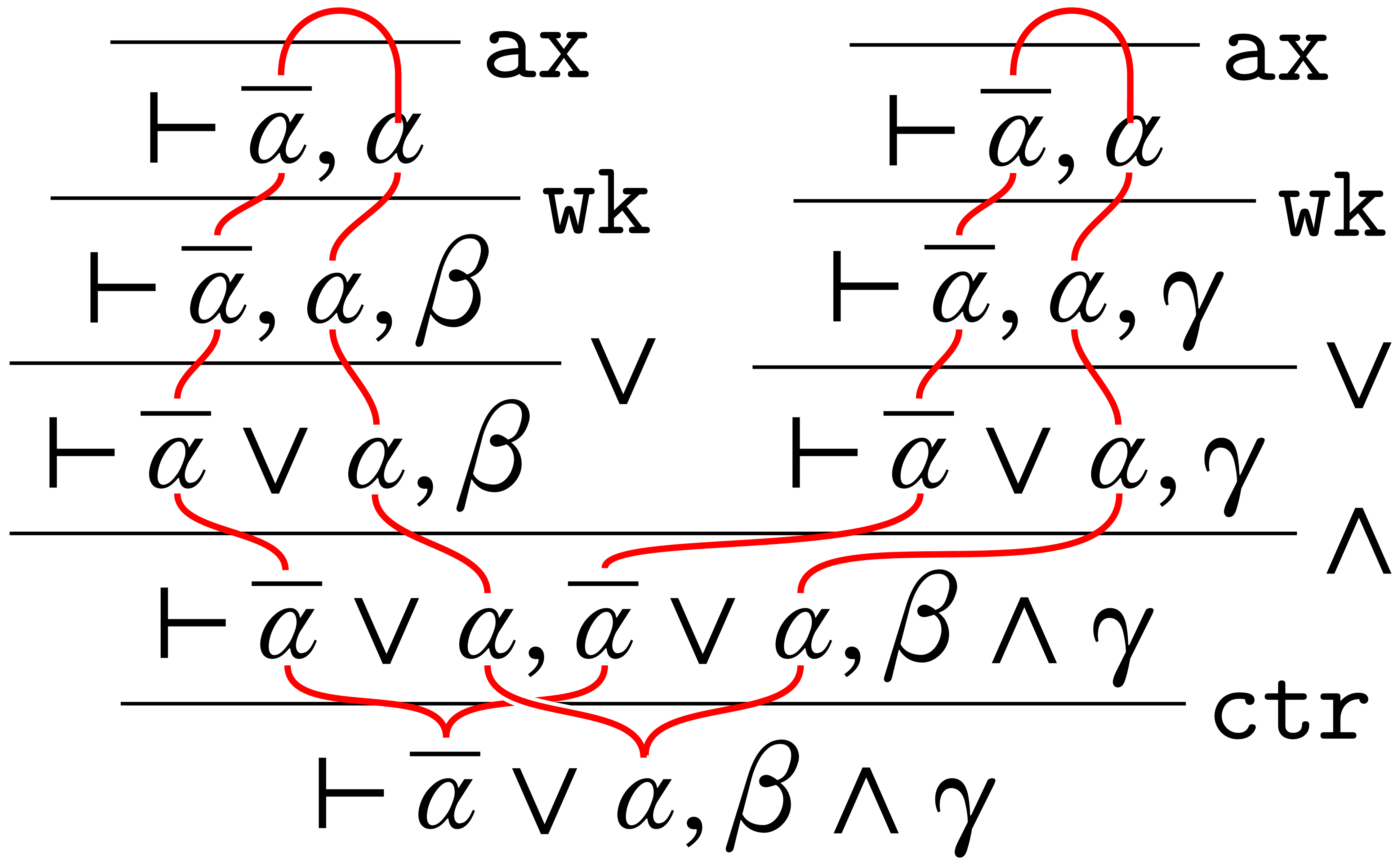
$$\dfrac{\dfrac{\overset{\vdots\,P}{\vdash \Gamma,\overline{F},\overline{G}} \quad \overset{\vdots\,Q}{\vdash \Delta_1,F}}{\vdash \Gamma,\Delta_1,\overline{G}}\,\texttt{cut} \quad \overset{\vdots\,R}{\vdash \Delta_2,G}}{\vdash \Gamma,\Delta_1,\Delta_2}\,\texttt{cut} \;\longleftarrow\; \dfrac{\dfrac{\overset{\vdots\,P}{\vdash \Gamma,\overline{F},\overline{G}}}{\vdash \Gamma,\overline{F}\vee\overline{G}}\,\vee \quad \dfrac{\overset{\vdots\,Q}{\vdash \Delta_1,F} \quad \overset{\vdots\,R}{\vdash \Delta_2,G}}{\vdash \Delta_1,\Delta_2,F\wedge G}\,\wedge}{\vdash \Gamma,\Delta_1,\Delta_2}\,\texttt{cut} \;\longrightarrow\; \dfrac{\dfrac{\overset{\vdots\,P}{\vdash \Gamma,\overline{F},\overline{G}} \quad \overset{\vdots\,R}{\vdash \Delta_2,G}}{\vdash \Gamma,\overline{F},\Delta_2}\,\texttt{cut} \quad \overset{\vdots\,Q}{\vdash \Delta_1,F}}{\vdash \Gamma,\Delta_1,\Delta_2}\,\texttt{cut}$$

$$\dfrac{\overset{\vdots\,P}{\vdash \Gamma}}{\vdash \Gamma,\Delta}\,\texttt{wk} \;\longleftarrow\; \dfrac{\dfrac{\overset{\vdots\,P}{\vdash \Gamma}}{\vdash \Gamma,F}\,\texttt{wk} \quad \dfrac{\overset{\vdots\,Q}{\vdash \Delta}}{\vdash \Delta,\overline{F}}\,\texttt{wk}}{\vdash \Gamma,\Delta}\,\texttt{cut} \;\longrightarrow\; \dfrac{\overset{\vdots\,Q}{\vdash \Delta}}{\vdash \Gamma,\Delta}\,\texttt{wk}$$

$$\dfrac{\dfrac{\dfrac{\overset{\vdots\,P}{\vdash \Gamma,F,F} \quad \dfrac{\overset{\vdots\,Q}{\vdash \Delta,\overline{F},\overline{F}}}{\vdash \Delta,\overline{F}}\,\texttt{ctr}}{\vdash \Gamma,\Delta}\,\texttt{cut} \quad \dfrac{\overset{\vdots\,Q}{\vdash \Delta,\overline{F},\overline{F}}}{\vdash \Delta,\overline{F}}\,\texttt{ctr}}{\vdash \Gamma,\Delta,\Delta}\,\texttt{cut}}{\vdash \Gamma,\Delta}\,\texttt{ctr} \;\longleftarrow\; \dfrac{\dfrac{\overset{\vdots\,P}{\vdash \Gamma,F,F}}{\vdash \Gamma,F}\,\texttt{ctr} \quad \dfrac{\overset{\vdots\,Q}{\vdash \Delta,\overline{F},\overline{F}}}{\vdash \Delta,\overline{F}}\,\texttt{ctr}}{\vdash \Gamma,\Delta}\,\texttt{cut} \;\longrightarrow\; \dfrac{\dfrac{\dfrac{\overset{\vdots\,P}{\vdash \Gamma,F,F}}{\vdash \Gamma,F}\,\texttt{ctr} \quad \dfrac{\dfrac{\overset{\vdots\,P}{\vdash \Gamma,F,F}}{\vdash \Gamma,F}\,\texttt{ctr} \quad \overset{\vdots\,Q}{\vdash \Delta,\overline{F},\overline{F}}}{\vdash \Gamma,\Delta}\,\texttt{cut}}{\vdash \Gamma,\Gamma}\,\texttt{cut}}{\vdash \Gamma,\Delta}\,\texttt{ctr}$$

IDEA #1 — TRACKING AXIOMS

Track existence, avoid counting.
(Andrews 1976, Lamarche & Straßburger 2004+)



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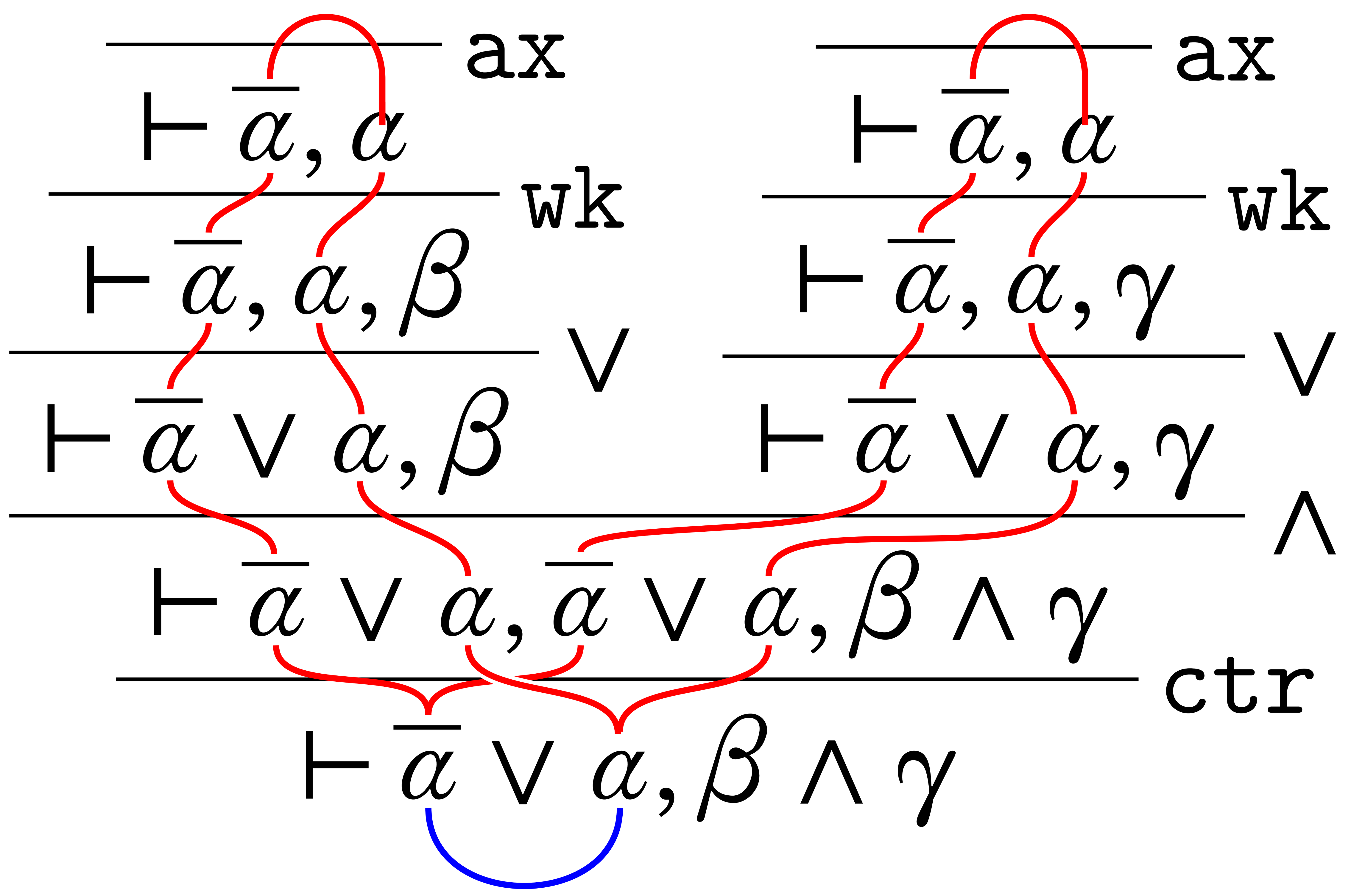

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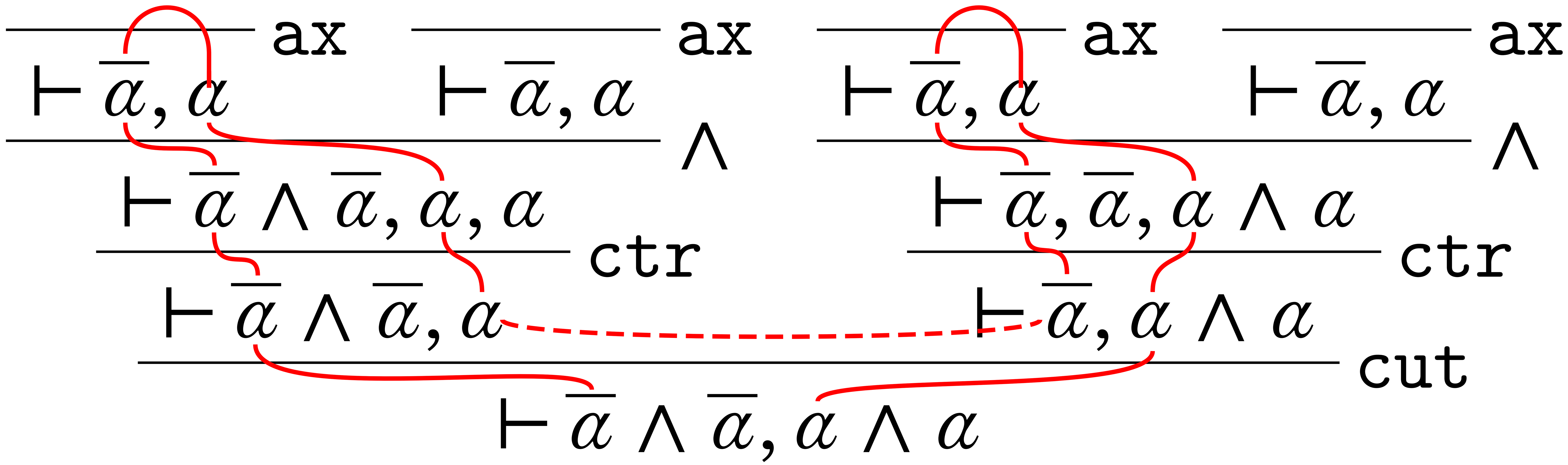

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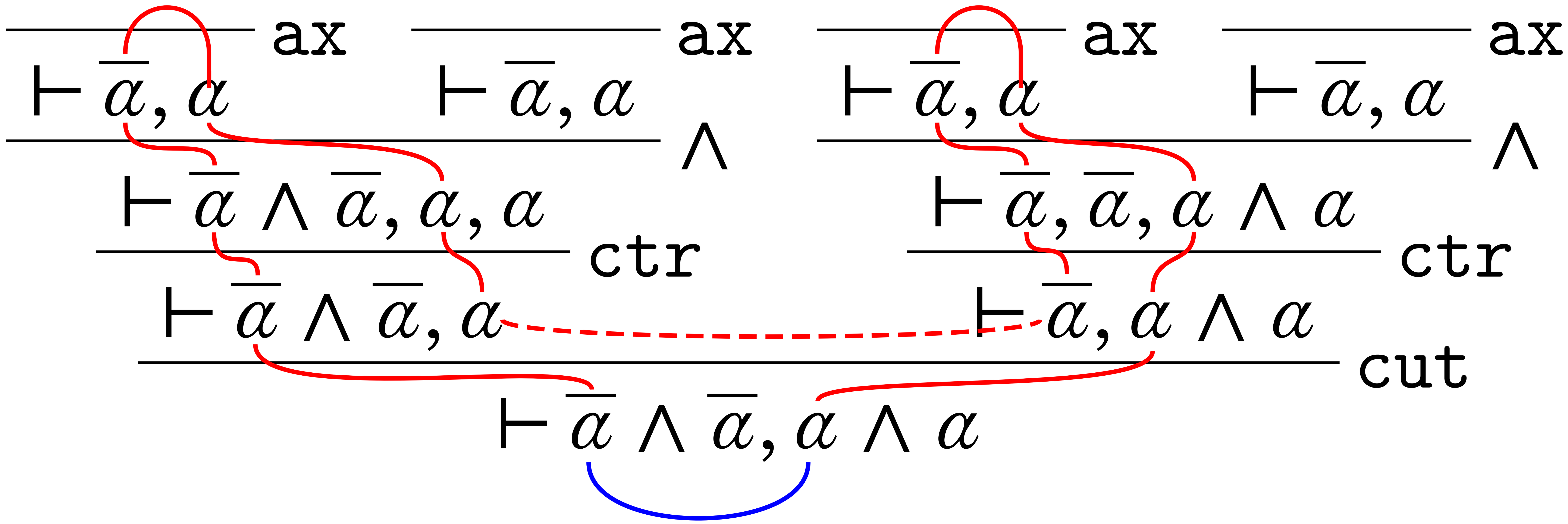

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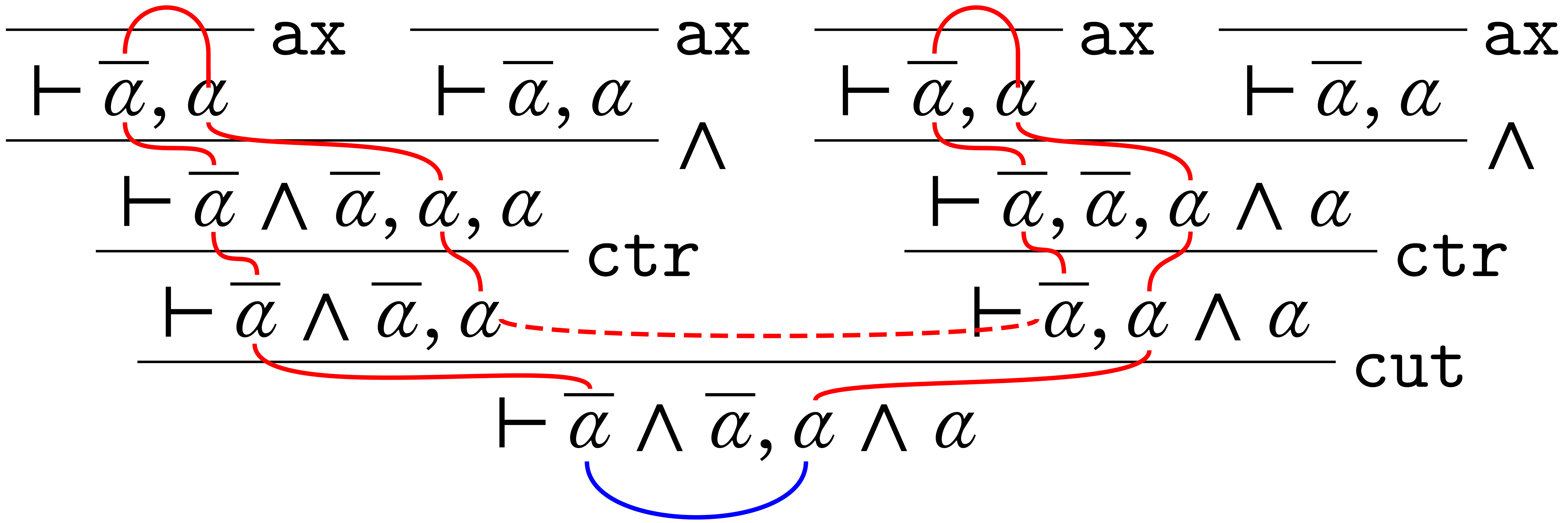
Theorem (Führmann & Pym 2004). *Mating graphs decrease under cut-reduction.*



IDEA #1 — TRACKING AXIOMS

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Axioms may be preserved, deleted,
duplicated, but *never created*.

$$\dfrac{\dfrac{\dfrac{}{\vdash \overline{F}, F}\,\texttt{ax} \qquad \dfrac{\begin{matrix}\vdots \\ \vdash G, H, H\end{matrix}}{\vdash G, H}\,\texttt{ctr}}{\vdash \overline{F}, F \wedge G, H}\,\wedge \qquad \dfrac{\begin{matrix}\vdots \\ \vdash \overline{H}, \overline{H}\end{matrix}}{\vdash \overline{H}}\,\texttt{ctr}}{\vdash \overline{F}, F \wedge G}\,\texttt{cut}$$

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$$\dfrac{\dfrac{}{\vdash \overline{F}, F}\,\texttt{ax} \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \vdash G, H, H\end{array}}{\vdash G, H}\,\texttt{ctr} \qquad \dfrac{\begin{array}{c}\vdots\\ \vdash \overline{H}, \overline{H}\end{array}}{\vdash \overline{H}}\,\texttt{ctr}}{\vdash G}\,\texttt{cut}}{\vdash \overline{F}, F \wedge G}\,\wedge$$

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\dfrac{}{\vdash \overline{F}, F}\,\texttt{ax}
\qquad
\dfrac{\vdots \atop \vdash G, H, H}{\vdash G, H}\,\texttt{ctr}
}{\vdash \overline{F}, F \wedge G, H}\,\wedge
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\dfrac{
\dfrac{}{\vdash \overline{F}, F}\,\texttt{ax}
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\dfrac{\vdots \atop \vdash G, H, H}{\vdash G, H}\,\texttt{ctr}
}{\vdash \overline{F}, F \wedge G, H}\,\wedge
\qquad
\dfrac{\vdots \atop \vdash \overline{H}, \overline{H}}{\vdash \overline{H}}\,\texttt{ctr}
}{\vdash \overline{F}, F \wedge G, \overline{H}}\,\texttt{cut}
}{
\dfrac{\vdash \overline{F}, F \wedge G, \overline{F}, F \wedge G}{\vdash \overline{F}, F \wedge G}\,\texttt{ctr}
}\,\texttt{cut}
$$

MATINGS ARE NOT INVARIANT UNDER DUPLICATION!

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\dfrac{\vdots\, P}{\vdash F} \quad \dfrac{\vdots\, Q}{\vdash G, F}
}{\vdash F \wedge G, F}\,\wedge
\qquad \dfrac{\vdots\, R}{\vdash G}
}{\vdash F \wedge G, F \wedge G}\,\wedge
}{\vdash F \wedge G}\,\mathtt{ctr}
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\dfrac{\dfrac{\dfrac{}{\vdash \overline{F}, F}\,\mathtt{ax}}{\vdash \overline{F}, \overline{G}, F}\,\mathtt{wk}}{\vdash \overline{F} \vee \overline{G}, F}\,\vee
\quad
\dfrac{\dfrac{\dfrac{}{\vdash \overline{G}, G}\,\mathtt{ax}}{\vdash \overline{F}, \overline{G}, G}\,\mathtt{wk}}{\vdash \overline{F} \vee \overline{G}, G}\,\vee
}{\vdash \overline{F} \vee \overline{G}, \overline{F} \vee \overline{G}, F \wedge G}\,\wedge
}{\vdash \overline{F} \vee \overline{G}, F \wedge G}\,\mathtt{ctr}
}{}
}{\vdash F \wedge G}\,\mathtt{cut}
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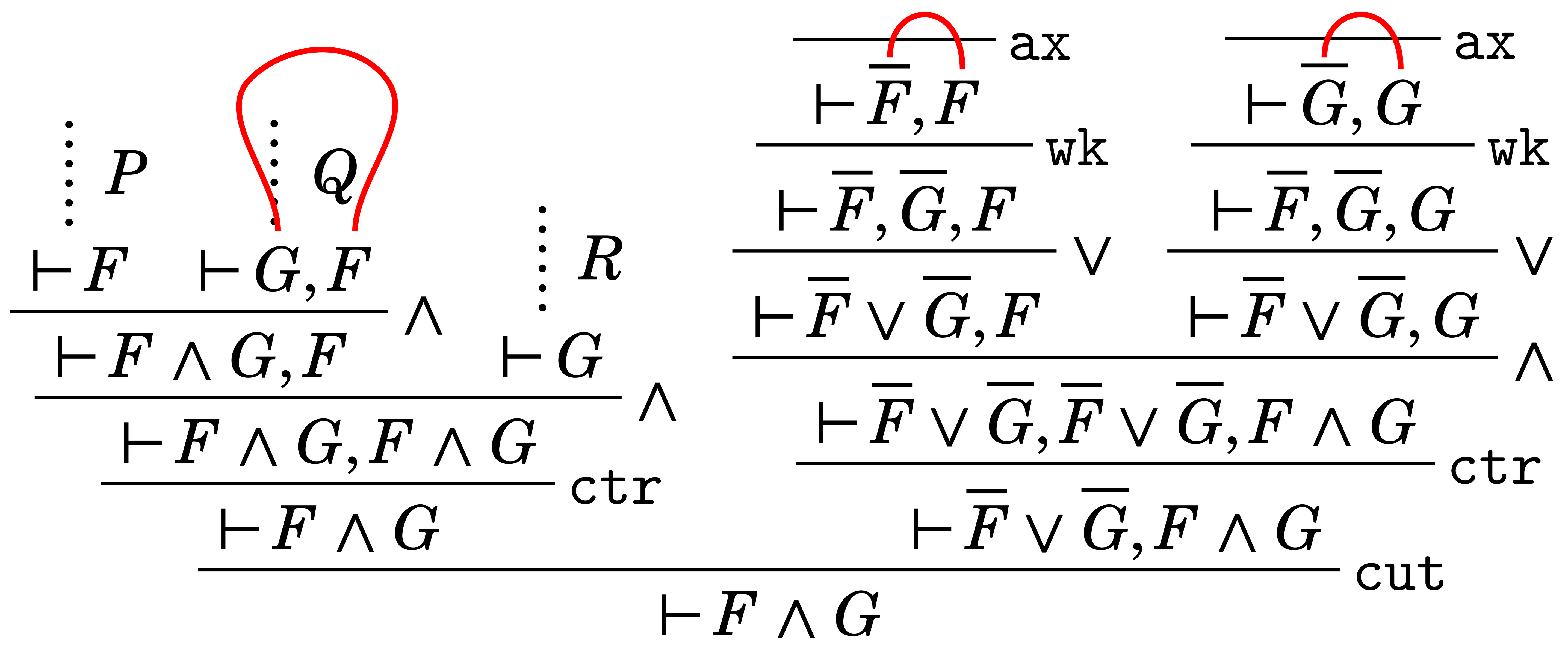
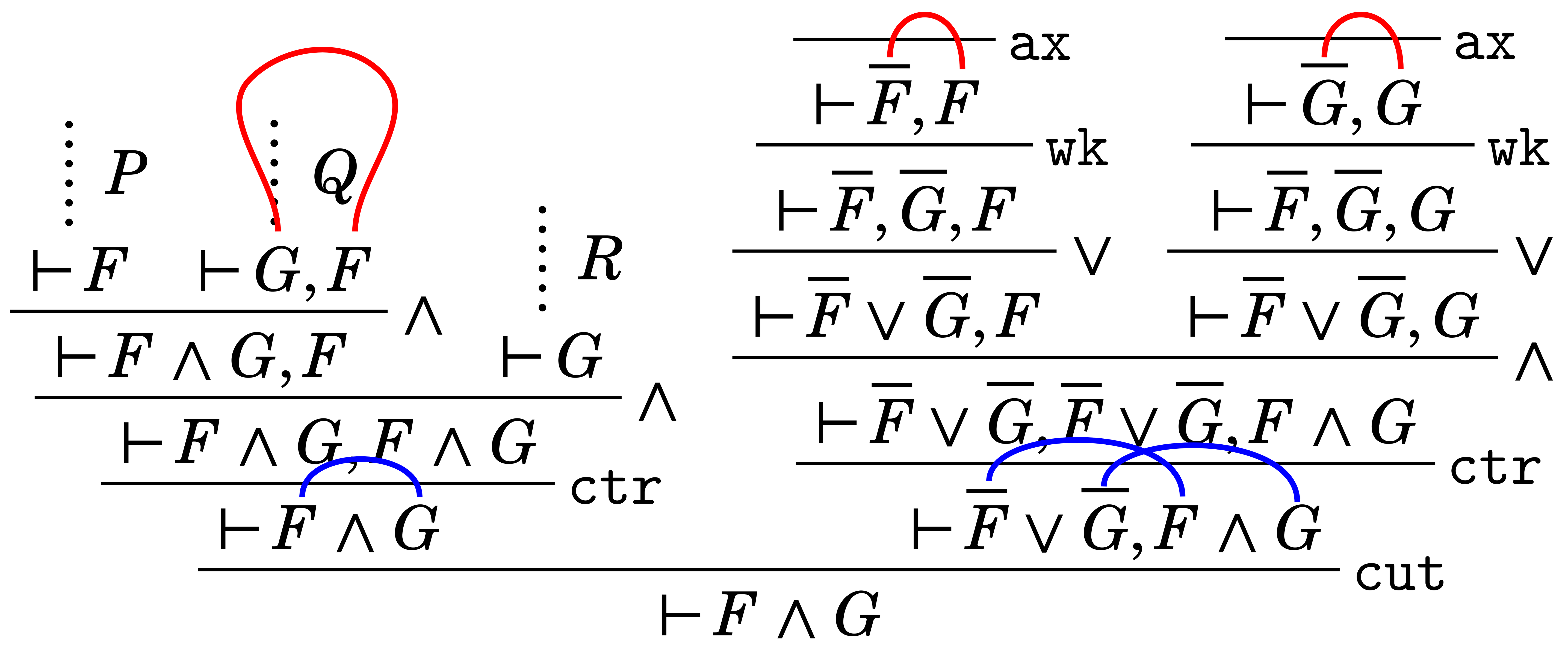


MATINGS ARE NOT INVARIANT UNDER DUPLICATION!



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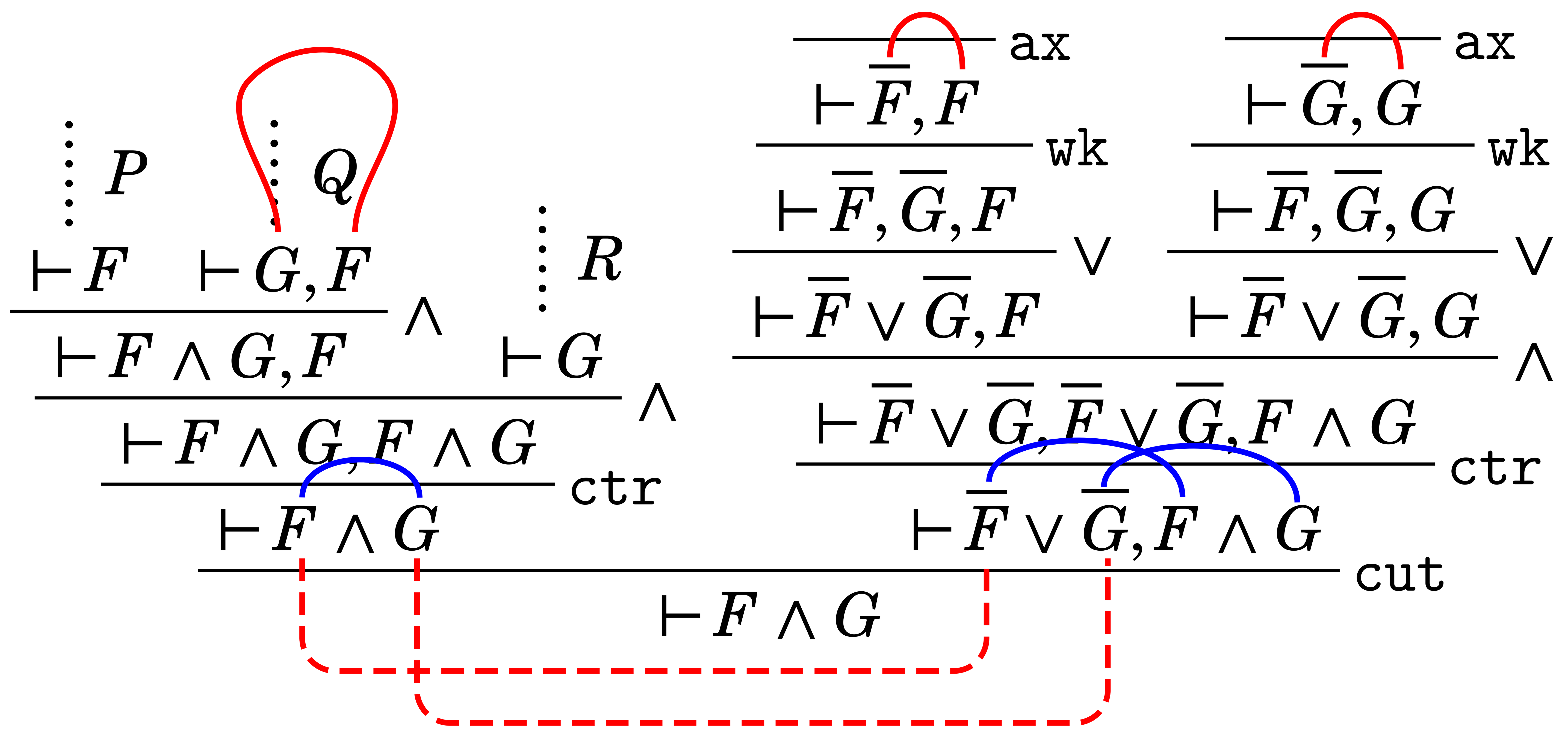

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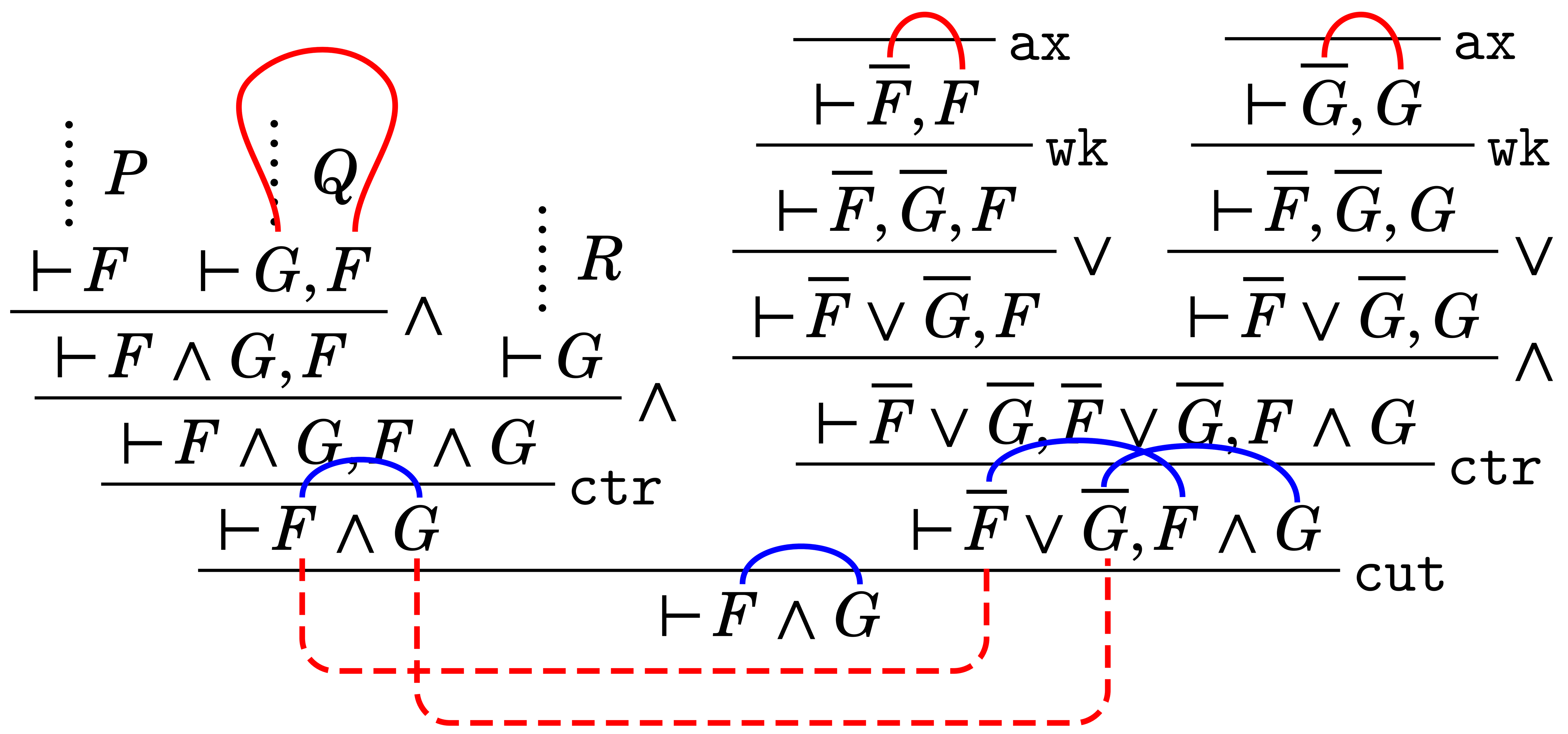

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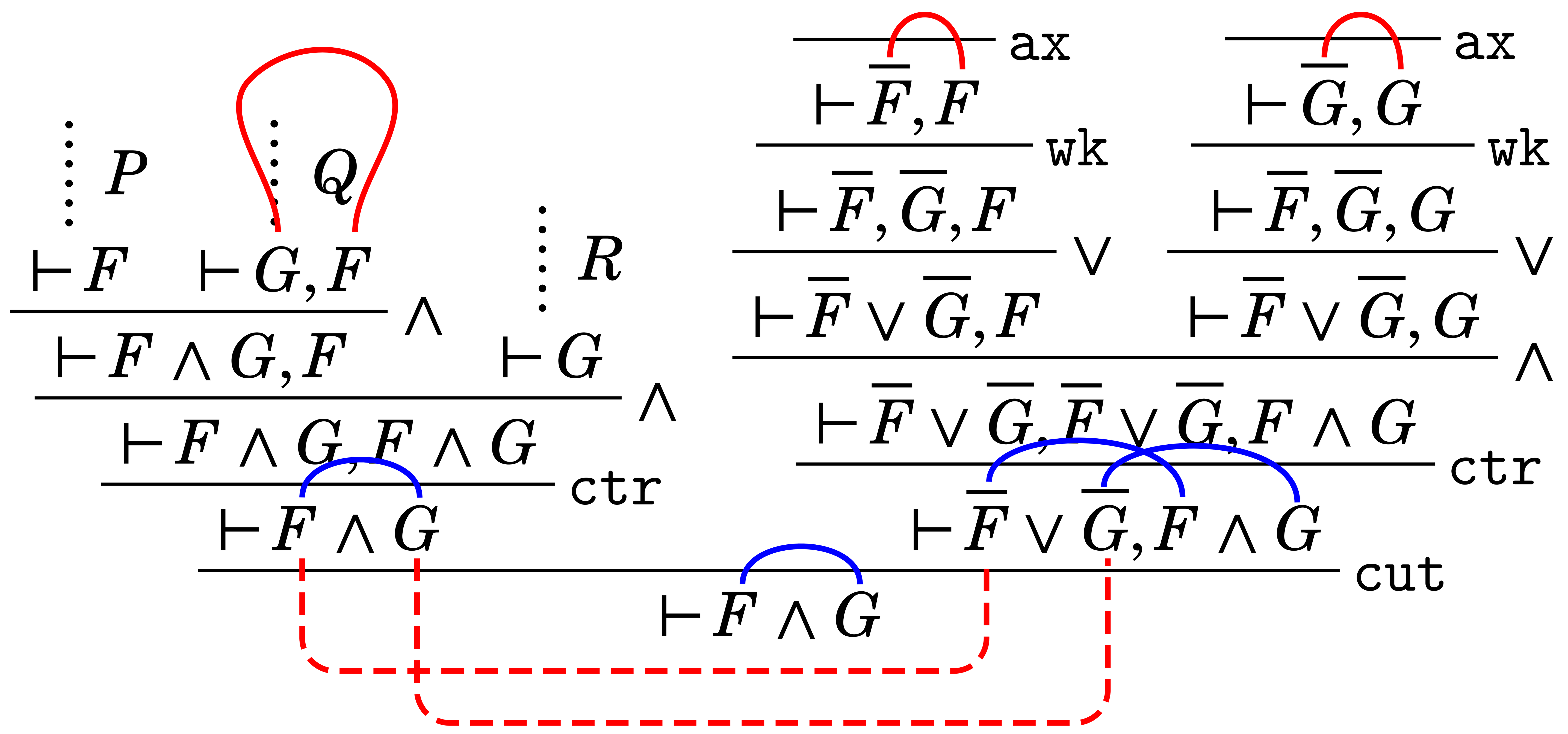
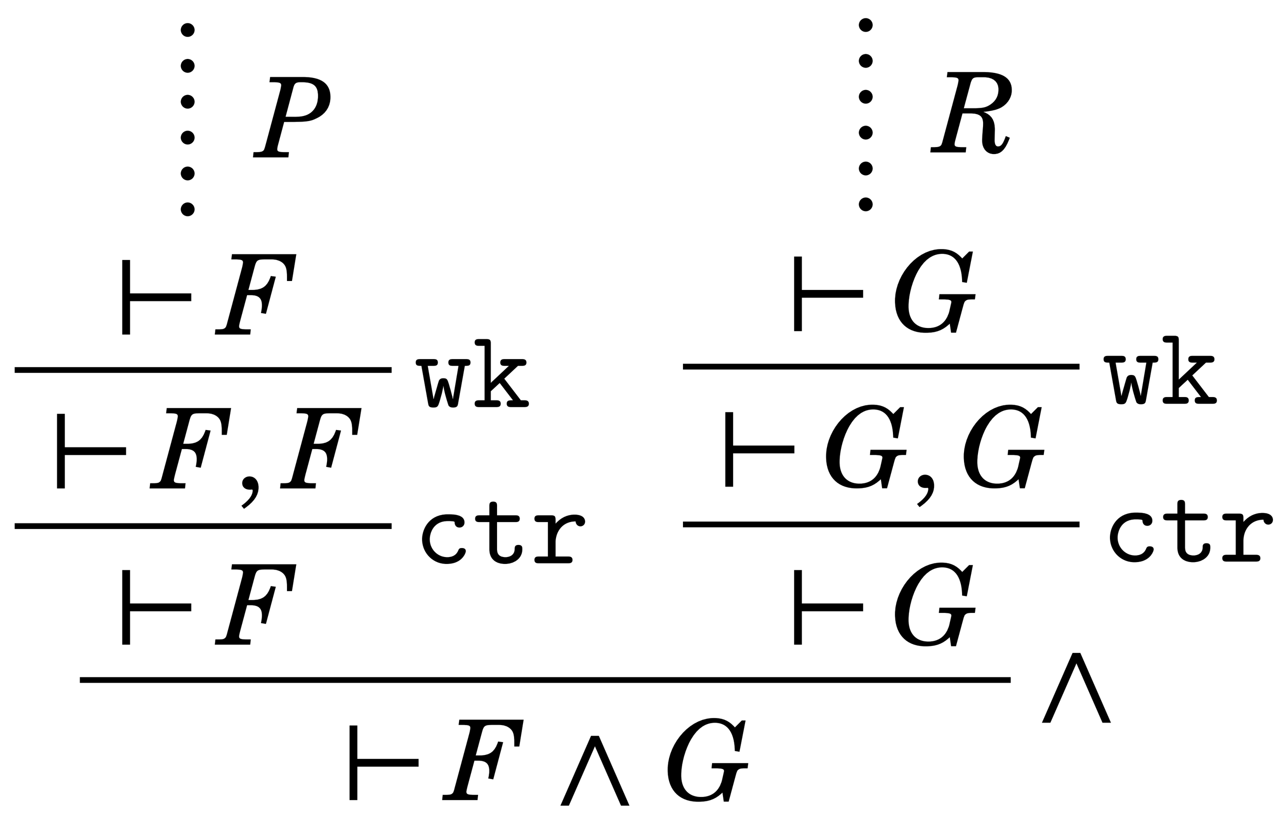




IDEA #2 — TARGET INVERTIBILITY

Identity group:

$$\frac{}{\vdash \Gamma, A, \overline{A}}\,\texttt{ax} \qquad \frac{\vdash \Gamma, A \quad \vdash \Gamma, \overline{A}}{\vdash \Gamma}\,\texttt{cut}$$

Structural group:

$$\frac{\vdash \Gamma \quad \vdash \Gamma}{\vdash \Gamma}\,\texttt{sum}$$

Logical group:

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}\,\vee \qquad \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\,\wedge$$

IDEA #2 — TARGET INVERTIBILITY

$$\dfrac{\dfrac{\vdash A, B, C \quad \vdash A, B, D}{\vdash A, B, C \wedge D}\wedge}{\vdash A \vee B, C \wedge D}\vee \quad \longleftrightarrow \quad \dfrac{\dfrac{\vdash A, B, C}{\vdash A \vee B, C}\vee \quad \dfrac{\vdash A, B, D}{\vdash A \vee B, D}\vee}{\vdash A \vee B, C \wedge D}\wedge$$

$$\dfrac{\dfrac{\vdash A, C \quad \vdash A, D}{\vdash A, C \wedge D}\wedge \quad \dfrac{\vdash B, C \quad \vdash B, D}{\vdash B, C \wedge D}\wedge}{\vdash A \wedge B, C \wedge D}\wedge \quad \longleftrightarrow \quad \dfrac{\dfrac{\vdash A, C \quad \vdash B, C}{\vdash A \wedge B, C}\wedge \quad \dfrac{\vdash A, D \quad \vdash B, D}{\vdash A \wedge B, D}\wedge}{\vdash A \wedge B, C \wedge D}\wedge$$

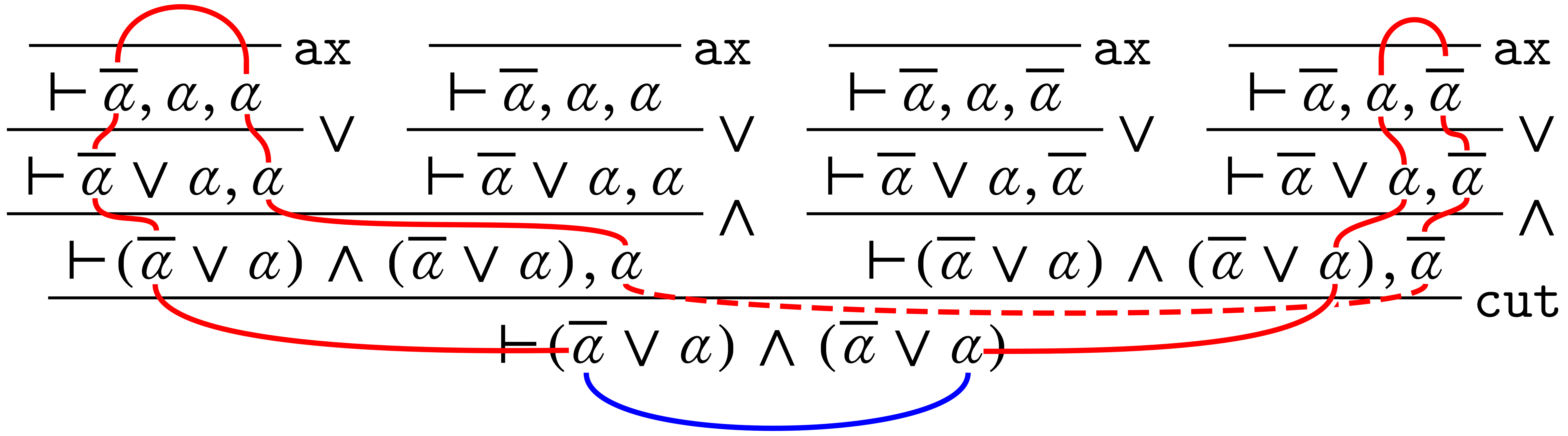
IDEA #2 — TARGET INVERTIBILITY

$$\dfrac{\dfrac{\vdash A, B, C \quad \vdash A, B, D}{\vdash A, B, C \wedge D}\wedge}{\vdash A \vee B, C \wedge D}\vee \quad \longleftrightarrow \quad \dfrac{\dfrac{\vdash A, B, C}{\vdash A \vee B, C}\vee \quad \dfrac{\vdash A, B, D}{\vdash A \vee B, D}\vee}{\vdash A \vee B, C \wedge D}\wedge$$

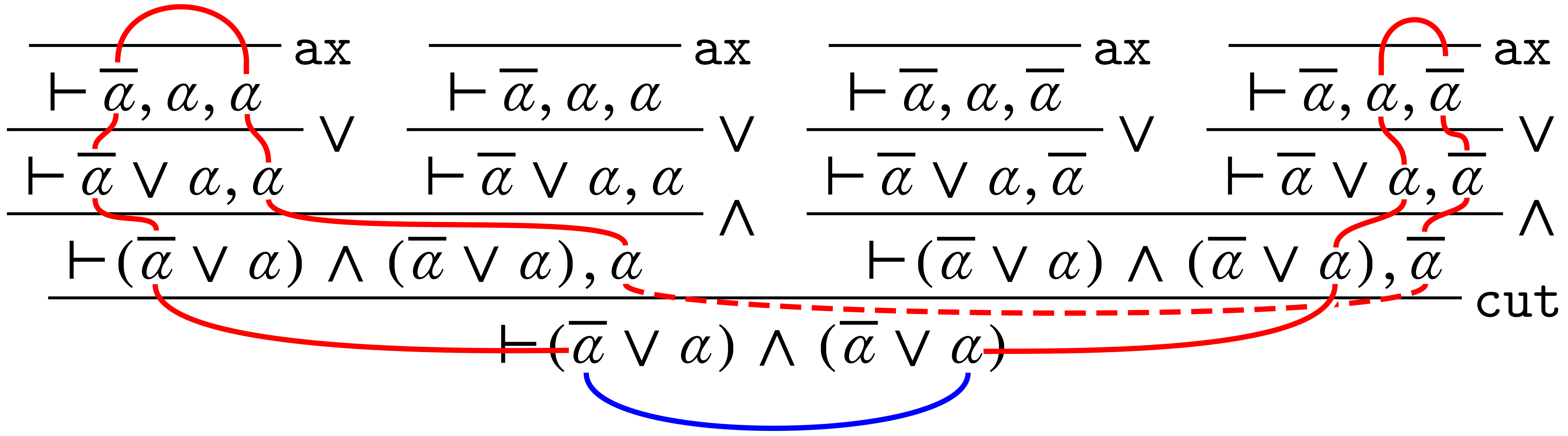
$$\dfrac{\dfrac{\vdash A, C \quad \vdash A, D}{\vdash A, C \wedge D}\wedge \quad \dfrac{\vdash B, C \quad \vdash B, D}{\vdash B, C \wedge D}\wedge}{\vdash A \wedge B, C \wedge D}\wedge \quad \longleftrightarrow \quad \dfrac{\dfrac{\vdash A, C \quad \vdash B, C}{\vdash A \wedge B, C}\wedge \quad \dfrac{\vdash A, D \quad \vdash B, D}{\vdash A \wedge B, D}\wedge}{\vdash A \wedge B, C \wedge D}\wedge$$

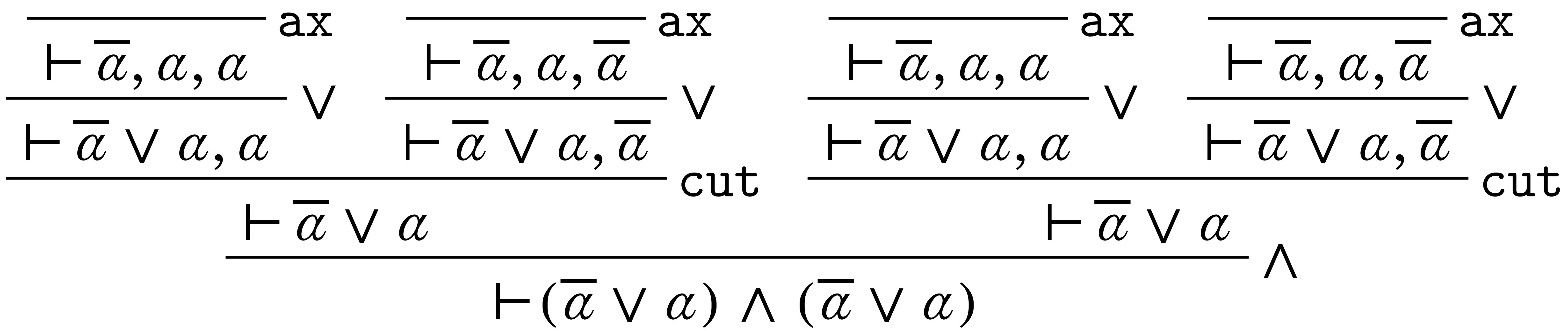
Theorem. *Mating graphs are invariant under arbitrary permutations of logical rules in the cut-free fragment of GS3.*

IDEA #3 — ADD BRANCH ANNOTATIONS



IDEA #3 — ADD BRANCH ANNOTATIONS





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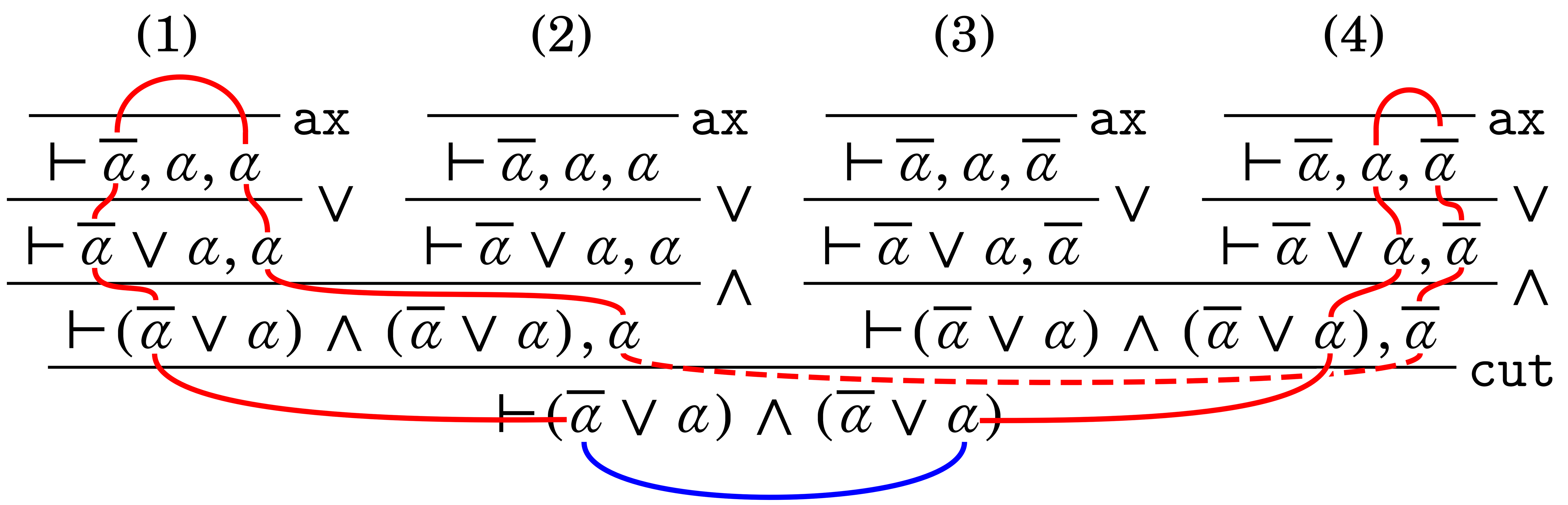
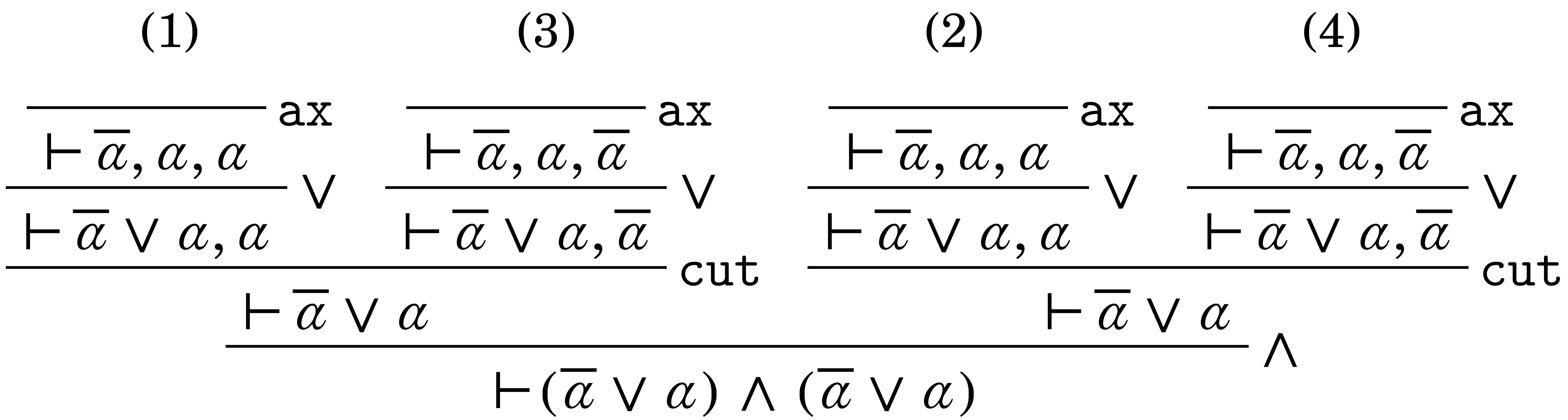




NAMED FORMULAS & SEQUENTS

$$A, B ::= \alpha_x \mid \overline{\alpha}_x \mid A \vee B \mid A \wedge B \qquad (\alpha \in \mathcal{A}, x \in \mathcal{N})$$

$$\overline{(\alpha_x)} = \overline{\alpha}_x \quad \overline{(\overline{\alpha}_x)} = \alpha_x \quad \overline{A \vee B} = \overline{A} \wedge \overline{B} \quad \overline{A \wedge B} = \overline{A} \vee \overline{B}$$ *(negation)*

$$|\alpha_x| = \alpha \quad |\overline{\alpha}_x| = \overline{\alpha} \quad |A \vee B| = |A| \wedge |B| \quad |A \wedge B| = |A| \vee |B|$$ *(underlying formula)*

$$A \equiv B \iff |A| = |B|$$ *(congruence)*

$$\vdash A_1, \ldots, A_n$$ *(sequent)*

Sharing-free formula, sequent, set of formulas: *each name occurs at most once.*

NAMED DERIVATIONS

Identity group:

$$\frac{}{\vdash \Gamma, A, \overline{B}}\ \mathtt{ax}_{\{A,\overline{B}\}} \quad (A \equiv B) \qquad\qquad \frac{\vdash \Gamma, A \quad \vdash \Gamma, \overline{A}}{\vdash \Gamma}\ \mathtt{cut}$$

Structural group:

$$\frac{\vdash \Gamma \quad \vdash \Gamma}{\vdash \Gamma}\ \sqcup$$

Logical group:

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}\ \vee \qquad\qquad \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\ \wedge$$

BRANCH-LABELED MATINGS

Pairs $\quad G\ (H, K, \ldots) = \langle V_G, \prec_G \rangle$

where $\quad V_G \subseteq \mathcal{N}$ is a set of names (*vertices*)

$\prec_G \subseteq \binom{V_G}{2} \times \mathscr{P}(V_G)$ is a relation between *unordered pairs* of vertices (*edges*) and arbitrary sets of vertices (*branch labels*),

such that $\quad e \prec_G X \implies e \subseteq X$

Operations:

$$G \sqcup H = \langle V_G \cup V_H, \prec_G \cup \prec_H \rangle$$

$$G{\upharpoonright}_X = \langle V_G \cap X, \{(e, Y) \in \prec_G \mid Y \subseteq X\} \rangle$$

Subgraph relation:

$$G \sqsubseteq H \iff V_G \subseteq V_H \wedge \prec_G \subseteq \prec_H$$

BRANCH-SENSITIVE COMPOSITION

Composition:

$$G \odot_I H = \langle (V_G \cup V_H) \setminus I, \prec \rangle \quad (I \subseteq \mathcal{N})$$

where for all $x \neq y \in (V_G \cup V_H) \setminus I$

$$xy \prec Z$$

if and only if there is a *complete alternating Z-labeled path*

$$z_1, \ldots, z_n$$

between G and H through the interface *I,*

such that $z_1 = x$ and $z_n = y$.

BRANCH-LABELED ALTERNATING PATH

Labeling modulo interfaces:

$$e \prec_G^I X \iff \exists Y.\, e \prec_G Y \wedge X = Y \setminus I$$

Alternating X-labeled path between G and H through the interface I:

a *finite* sequence $x_1, \dots, x_n \in V_G \cup V_H$ of *pairwise distinct* vertices, such that

(i) $x_i \in I$ for all $1 < i < n$ (internal vertices lie in the interface I)

(ii) either $e_i \prec_G^I X$ for all odd $1 \le i < n$ and $e_i \prec_H^I X$ for all even $1 \le i < n$,
or $e_i \prec_H^I X$ for all odd $1 \le i < n$ and $e_i \prec_G^I X$ for all even $1 \le i < n$,

where $e_i = x_i x_{i+1}$.

Complete if $x_1, x_n \notin I$ (i.e. the endpoints lie outside the interface).

THE INTERPRETATION

$$\left[\!\!\left[\frac{}{\vdash \Gamma, \alpha_x, \overline{\alpha}_y}\,\mathsf{ax}_{\{\alpha_x, \overline{\alpha}_y\}} \right]\!\!\right] = \langle \mathsf{names}(\Gamma, \alpha_x, \overline{\alpha}_y), \{xy \prec \mathsf{names}(\Gamma, \alpha_x, \overline{\alpha}_y)\}\rangle \quad (\Gamma \text{ atomic})$$

$$\left[\!\!\left[\dfrac{\begin{array}{c}\vdots\, P \\ \vdash \Gamma, A\end{array} \quad \begin{array}{c}\vdots\, Q \\ \vdash \Gamma, \overline{A}\end{array}}{\vdash \Gamma}\,\mathsf{cut} \right]\!\!\right] = [\![P]\!] \odot_A [\![Q]\!]$$

$$\left[\!\!\left[\dfrac{\begin{array}{c}\vdots\, P_1 \\ \vdash \Gamma_1\end{array} \quad \dots \quad \begin{array}{c}\vdots\, P_n \\ \vdash \Gamma_n\end{array}}{\vdash \Gamma}\,\mathsf{rule} \right]\!\!\right] = [\![P_1]\!] \sqcup \dots \sqcup [\![P_n]\!] \quad (\mathsf{rule} \in \{\sqcup, \vee, \wedge\})$$

MAIN RESULTS

Theorem (Soundness). *Branch-annotated mating graphs are invariant under arbitrary permutations of logical rules in the full calculus.*

Proof sketch

1. Use compositionality.

2. Trivial for logical, union rule commutations.

3. For commutations involving cuts, reason about alternating paths.
The only complex case is the conjunction-cut one.

MAIN RESULTS

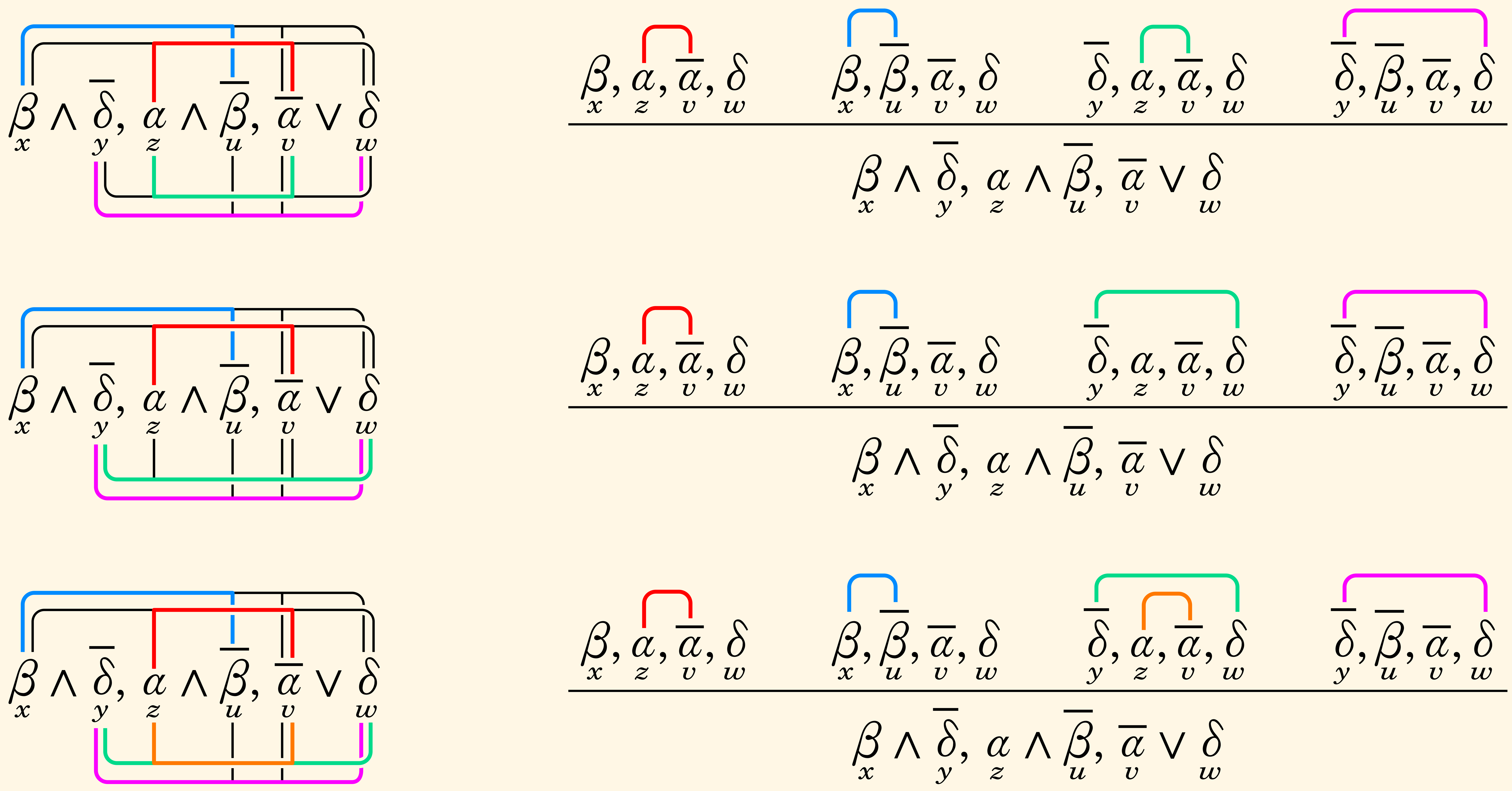
Theorem (Cut-elimination). *For any GS4 derivation there is a cut-free GS4 derivation with the same conclusion and denotation.*

Proof sketch

1. Upon finding a cut, normalize recursively its sub-derivations.

2. Use logical rule permutations to reduce the *conclusion* of the cut to an atomic clause.

3. Compute the denotation and reconstruct the resulting derivation.



MAIN RESULTS

Theorem (Sequentialization). *A branch-annotated mating G is correct w.r.t. some sequent* $\vdash \Gamma$ *if and only if there is a GS4 derivation* $P \vdash \Gamma$ *whose denotation is G*

Moreover, the size of P is polynomially bounded by the size of G.

MAIN RESULTS

Theorem (Sequentialization). *A branch-annotated mating G is correct w.r.t. some sequent* $\vdash \Gamma$ *if and only if there is a GS4 derivation* $P \vdash \Gamma$ *whose denotation is G*

Moreover, the size of P is polynomially bounded by the size of G.

Rule permutations are identities

Efficient invertibility

Admissibility of rules by algebraic reasoning

SHORTCOMINGS

- Cuts do not commute!
- No local cut-reduction procedure.
- Has the original problem been solved? Unclear.
- No predicate calculus.

THANK YOU