

**ON THE SEMANTICS OF PROOFS  
IN CLASSICAL SEQUENT CALCULUS**

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# GENTZEN'S CALCULUS LK (PROPOSITIONAL FRAGMENT)

$$F, G ::= \alpha \mid \neg F \mid F \rightarrow G \mid F \vee G \mid F \wedge G \quad (\alpha \in \mathcal{A})$$

*Identity group:*

$$\frac{}{F \vdash F} \text{ ax} \quad \frac{\Gamma \vdash \Delta, F \quad F, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{ cut}$$

*Structural group:*

$$\frac{\Gamma, F, G, \Gamma' \vdash \Delta}{\Gamma, G, F, \Gamma' \vdash \Delta} \text{ xch} \vdash \quad \frac{\Gamma \vdash \Delta}{F, \Gamma \vdash \Delta} \text{ wk} \vdash \quad \frac{F, F, \Gamma \vdash \Delta}{F, \Gamma \vdash \Delta} \text{ ctr} \vdash$$

$$\frac{\Gamma \vdash \Delta, F, G \Delta'}{\Gamma \vdash \Delta, G, F, \Delta'} \vdash \text{ xch} \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, F} \vdash \text{ wk} \quad \frac{\Gamma \vdash \Delta, F, F}{\Gamma \vdash \Delta, F, F} \vdash \text{ ctr}$$

*Additive group (context sharing):*

$$\frac{F, \Gamma \vdash \Delta}{F \wedge G, \Gamma \vdash \Delta} \wedge_1 \vdash \quad \frac{\Gamma \vdash \Delta, F}{\Gamma \vdash \Delta, F \vee G} \vee_1 \vdash$$

$$\frac{G, \Gamma \vdash \Delta}{F \wedge G, \Gamma \vdash \Delta} \wedge_r \vdash \quad \frac{\Gamma \vdash \Delta, G}{\Gamma \vdash \Delta, F \vee G} \vee_r \vdash$$

$$\frac{\Gamma \vdash \Delta, F \quad \Gamma \vdash \Delta, G}{\Gamma \vdash \Delta, F \wedge G} \vdash \wedge \quad \frac{F, \Gamma \vdash \Delta \quad G, \Gamma \vdash \Delta}{F \vee G, \Gamma \vdash \Delta} \vee \vdash$$

*Multiplicative group (context splitting):*

$$\frac{\Gamma \vdash \Delta, F}{\neg F, \Gamma \vdash \Delta} \neg \vdash \quad \frac{\Gamma \vdash \Delta, F \quad G, \Gamma' \vdash \Delta'}{F \rightarrow G, \Gamma, \Gamma' \vdash \Delta, \Delta'} \rightarrow \vdash$$

$$\frac{F, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg F} \vdash \neg \quad \frac{F, \Gamma \vdash \Delta, G}{\Gamma \vdash \Delta, F \rightarrow G} \vdash \rightarrow$$

# THE CALCULUS LK (PROPOSITIONAL FRAGMENT, ONE SIDED)

$$F, G ::= \alpha \mid \bar{\alpha} \mid F \vee G \mid F \wedge G \quad (\alpha \in \mathcal{A})$$

$$\overline{(\bar{\alpha})} = \bar{\alpha} \quad \overline{(\alpha)} = \alpha \quad \overline{F \vee G} = \bar{F} \wedge \bar{G} \quad \overline{F \wedge G} = \bar{F} \vee \bar{G}$$

*Identity group:*

$$\frac{}{\vdash \bar{A}, A} \text{ax} \quad \frac{\vdash \Gamma, A \quad \vdash \Delta, \bar{A}}{\vdash \Gamma, \Delta} \text{cut}$$

*Structural group:*

$$\frac{\vdash \Gamma}{\vdash \Gamma, A} \text{wk} \quad \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A} \text{ctr}$$

*Logical group:*

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B} \vee \quad \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \wedge B} \wedge$$

# CUT-ELIMINATION THEOREMS

**Coarse form.** *The cut-rule is admissible.*

**Refined form.** *For every derivation  $P \vdash \Gamma$  there is a cut-free derivation  $Q \vdash \Gamma$  such that  $P \longrightarrow^* Q$ .*

## **Denotational semantics:**

Define functions  $\llbracket - \rrbracket : \text{LK} \rightarrow X$  mapping derivations to *denotations*, such that

$$P \longrightarrow Q \implies \llbracket P \rrbracket = \llbracket Q \rrbracket$$

# CUT-ELIMINATION THEOREMS

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***Interesting invariants:***

*Coarsest* – correctness and conclusions.

*Finest* – normal form (if unique).

# CUT-ELIMINATION THEOREMS

**Coarse form.** *There is a rewriting relation that preserves correctness and conclusions and decreases the number of cuts.*

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# CUT-ELIMINATION THEOREMS

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***Interesting invariants:***

*Coarsest* – correctness and conclusions.

*Finest* – normal form (if unique).

*Intermediate ones?*

## CUT-ELIMINATION – KEY CASES

$$\frac{\frac{\frac{\vdots P}{\vdash \Gamma, \bar{F}, \bar{G}} \vee}{\vdash \Gamma, \bar{F} \vee \bar{G}} \quad \frac{\frac{\frac{\vdots Q \quad \vdots R}{\vdash \Delta_1, F \quad \vdash \Delta_2, G} \wedge}{\vdash \Delta_1, \Delta_2, F \wedge G} \wedge}{\vdash \Gamma, \Delta_1, \Delta_2} \text{cut}}{\vdash \Gamma, \Delta_1, \Delta_2} \text{cut} \quad \rightarrow \quad \frac{\frac{\frac{\frac{\vdots P \quad \vdots Q}{\vdash \Gamma, \bar{F}, \bar{G}} \quad \vdash \Delta_1, F}{\vdash \Gamma, \Delta_1, \bar{G}} \text{cut}}{\vdash \Gamma, \Delta_1, \Delta_2} \text{cut} \quad \frac{\vdots R}{\vdash \Delta_2, G}}{\vdash \Gamma, \Delta_1, \Delta_2} \text{cut}}{\vdash \Gamma, \Delta_1, \Delta_2} \text{cut}$$

$$\frac{\frac{\frac{\frac{\vdots P}{\vdash \Gamma, \bar{F}, \bar{G}} \vee}{\vdash \Gamma, \bar{F} \vee \bar{G}} \quad \frac{\frac{\frac{\vdots Q \quad \vdots R}{\vdash \Delta_1, F \quad \vdash \Delta_2, G} \wedge}{\vdash \Delta_1, \Delta_2, F \wedge G} \wedge}{\vdash \Gamma, \Delta_1, \Delta_2} \text{cut}}{\vdash \Gamma, \Delta_1, \Delta_2} \text{cut} \quad \rightarrow \quad \frac{\frac{\frac{\frac{\vdots P \quad \vdots R}{\vdash \Gamma, \bar{F}, \bar{G}} \quad \vdash \Delta_2, G}{\vdash \Gamma, \bar{F}, \Delta_2} \text{cut}}{\vdash \Gamma, \Delta_1, \Delta_2} \text{cut} \quad \frac{\vdots Q}{\vdash \Delta_1, F}}{\vdash \Gamma, \Delta_1, \Delta_2} \text{cut}}{\vdash \Gamma, \Delta_1, \Delta_2} \text{cut}$$



# CUT-ELIMINATION – STRUCTURAL CASES

$$\frac{\frac{\vdots P}{\vdash \Gamma} \text{wk} \quad \frac{\vdots Q}{\vdash \Delta, \bar{F}} \text{cut}}{\vdash \Gamma, \Delta} \text{cut}$$

→

$$\frac{\frac{\vdots P}{\vdash \Gamma} \text{wk}}{\vdash \Gamma, \Delta} \text{wk}$$

$$\frac{\frac{\frac{\vdots P}{\vdash \Gamma, F, F} \text{ctr} \quad \frac{\vdots Q}{\vdash \Delta, \bar{F}} \text{cut}}{\vdash \Gamma, F} \text{ctr}}{\vdash \Gamma, \Delta} \text{cut}$$

→

$$\frac{\frac{\frac{\frac{\vdots P}{\vdash \Gamma, F, F} \text{ctr} \quad \frac{\vdots Q}{\vdash \Delta, \bar{F}} \text{cut}}{\vdash \Gamma, \Delta} \text{cut} \quad \frac{\vdots Q}{\vdash \Delta, \bar{F}} \text{cut}}{\frac{\vdash \Gamma, \Delta, \Delta}{\vdash \Gamma, \Delta} \text{ctr}} \text{cut}$$

## CUT-ELIMINATION – COMMUTATIVE CASES (EXAMPLES)

$$\frac{\frac{\frac{\vdots P}{\vdots} \quad \frac{\vdots Q}{\vdots}}{\vdash \Gamma, F \vee G, H} \vee \quad \frac{\vdots Q}{\vdash \Delta, \bar{H}}}{\vdash \Gamma, F \vee G, \Delta} \text{cut} \quad \longrightarrow \quad \frac{\frac{\frac{\vdots P}{\vdots} \quad \frac{\vdots Q}{\vdots}}{\vdash \Gamma, F, G, H \quad \vdash \Delta, \bar{H}} \text{cut}}{\vdash \Gamma, F \vee G, \Delta} \vee$$

$$\frac{\frac{\frac{\vdots P}{\vdots} \quad \frac{\vdots Q}{\vdots}}{\vdash \Gamma_1, F, H \quad \vdash \Gamma_2, G} \wedge \quad \frac{\vdots R}{\vdash \Delta, \bar{H}}}{\vdash \Gamma_1, \Gamma_2, F \wedge G, \Delta} \text{cut} \quad \longrightarrow \quad \frac{\frac{\frac{\vdots P}{\vdots} \quad \frac{\vdots R}{\vdots}}{\vdash \Gamma_1, F, H \quad \vdash \Delta, \bar{H}} \text{cut} \quad \frac{\vdots Q}{\vdash \Gamma_2, G}}{\vdash \Gamma_1, \Gamma_2, F \wedge G, \Delta} \wedge$$

## PATHOLOGICAL CRITICAL PAIRS

$$\begin{array}{c}
 \begin{array}{c} \vdots P \\ \vdots Q \\ \hline \vdots P \quad \vdots Q \\ \vdash \Gamma, \bar{F}, \bar{G} \quad \vdash \Delta_1, F \end{array} \text{cut} \\
 \hline \vdash \Gamma, \Delta_1, \bar{G} \\
 \hline \vdash \Gamma, \Delta_1, \Delta_2
 \end{array} \text{cut} \quad \vdash \Delta_2, G \quad \vdots R \\
 \leftarrow \frac{\begin{array}{c} \vdots P \\ \vdots Q \quad \vdots R \\ \hline \vdots P \quad \vdots Q \quad \vdots R \\ \vdash \Gamma, \bar{F}, \bar{G} \quad \vdash \Delta_1, F \quad \vdash \Delta_2, G \end{array} \vee \quad \wedge \\
 \hline \vdash \Gamma, \bar{F} \vee \bar{G} \quad \vdash \Delta_1, \Delta_2, F \wedge G \\
 \hline \vdash \Gamma, \Delta_1, \Delta_2 \text{cut}}{\vdash \Gamma, \Delta_1, \Delta_2} \\
 \rightarrow \frac{\begin{array}{c} \vdots P \quad \vdots R \\ \hline \vdots P \quad \vdots R \\ \vdash \Gamma, \bar{F}, \bar{G} \quad \vdash \Delta_2, G \end{array} \text{cut} \quad \vdots Q \\
 \hline \vdash \Gamma, \bar{F}, \Delta_2 \quad \vdash \Delta_1, F \\
 \hline \vdash \Gamma, \Delta_1, \Delta_2 \text{cut}}{\vdash \Gamma, \Delta_1, \Delta_2}
 \end{array}$$

$$\begin{array}{c} \vdots P \\ \vdash \Gamma \\ \hline \vdash \Gamma, \Delta \text{wk} \end{array} \leftarrow \frac{\begin{array}{c} \vdots P \quad \vdots Q \\ \hline \vdots P \quad \vdots Q \\ \vdash \Gamma \quad \vdash \Delta \\ \vdash \Gamma, F \quad \vdash \Delta, \bar{F} \end{array} \text{wk} \quad \wedge \\
 \hline \vdash \Gamma, \Delta \text{cut}}{\vdash \Gamma, \Delta} \rightarrow \frac{\begin{array}{c} \vdots Q \\ \hline \vdots Q \\ \vdash \Delta \\ \vdash \Gamma, \Delta \end{array} \text{wk}}{\vdash \Gamma, \Delta}$$

$$\begin{array}{c} \vdots Q \\ \vdash \Delta, \bar{F}, \bar{F} \\ \hline \vdots P \quad \vdots Q \\ \vdash \Gamma, F, F \quad \vdash \Delta, \bar{F} \end{array} \text{ctr} \quad \vdots Q \\
 \hline \vdash \Gamma, \Delta \quad \vdash \Delta, \bar{F} \\
 \hline \vdash \Gamma, \Delta, \Delta \\
 \hline \vdash \Gamma, \Delta \text{ctr} \\
 \leftarrow \frac{\begin{array}{c} \vdots P \quad \vdots Q \\ \hline \vdots P \quad \vdots Q \\ \vdash \Gamma, F, F \quad \vdash \Delta, \bar{F}, \bar{F} \end{array} \text{ctr} \quad \vdots Q \\
 \hline \vdash \Gamma, F \quad \vdash \Delta, \bar{F} \\
 \hline \vdash \Gamma, \Delta \text{cut}}{\vdash \Gamma, \Delta} \rightarrow \frac{\begin{array}{c} \vdots P \quad \vdots Q \\ \hline \vdots P \quad \vdots Q \\ \vdash \Gamma, F, F \quad \vdash \Delta, \bar{F}, \bar{F} \end{array} \text{ctr} \quad \vdots Q \\
 \hline \vdash \Gamma, F \quad \vdash \Gamma, \Delta \\
 \hline \vdash \Gamma, \Gamma \\
 \hline \vdash \Gamma, \Delta \text{ctr}}{\vdash \Gamma, \Delta}
 \end{array}$$

# PATHOLOGICAL CRITICAL PAIRS

$$\begin{array}{c}
 \begin{array}{c} \vdots P \\ \vdots Q \\ \hline \vdash \Gamma, \bar{F}, \bar{G} \quad \vdash \Delta_1, F \end{array} \text{cut} \\
 \hline
 \vdash \Gamma, \Delta_1, \bar{G} \\
 \hline
 \vdash \Gamma, \Delta_1, \Delta_2 \\
 \text{cut}
 \end{array}
 \leftarrow
 \begin{array}{c}
 \begin{array}{c} \vdots P \\ \vdots Q \\ \vdots R \\ \hline \vdash \Gamma, \bar{F}, \bar{G} \end{array} \vee \quad \begin{array}{c} \vdots Q \\ \vdots R \\ \hline \vdash \Delta_1, F \quad \vdash \Delta_2, G \end{array} \wedge \\
 \hline
 \vdash \Gamma, \bar{F} \vee \bar{G} \quad \vdash \Delta_1, \Delta_2, F \wedge G \\
 \hline
 \vdash \Gamma, \Delta_1, \Delta_2 \\
 \text{cut}
 \end{array}
 \rightarrow
 \begin{array}{c}
 \begin{array}{c} \vdots P \\ \vdots R \\ \hline \vdash \Gamma, \bar{F}, \bar{G} \quad \vdash \Delta_2, G \end{array} \text{cut} \\
 \hline
 \vdash \Gamma, \bar{F}, \Delta_2 \\
 \hline
 \vdash \Gamma, \Delta_1, \Delta_2 \\
 \text{cut}
 \end{array}
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{c} \vdots P \\ \hline \vdash \Gamma \\
 \hline \vdash \Gamma, \Delta \\
 \text{wk}
 \end{array}
 \leftarrow
 \begin{array}{c}
 \begin{array}{c} \vdots P \\ \vdots Q \\ \hline \vdash \Gamma \\ \hline \vdash \Gamma, F \end{array} \text{wk} \quad \begin{array}{c} \vdots Q \\ \hline \vdash \Delta \\ \hline \vdash \Delta, \bar{F} \end{array} \text{wk} \\
 \hline
 \vdash \Gamma, \Delta \\
 \text{cut}
 \end{array}
 \rightarrow
 \begin{array}{c}
 \begin{array}{c} \vdots Q \\ \hline \vdash \Delta \\
 \hline \vdash \Gamma, \Delta \\
 \text{wk}
 \end{array}
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{c} \vdots P \\ \vdots Q \\ \hline \vdash \Gamma, F, F \quad \vdash \Delta, \bar{F}, \bar{F} \end{array} \text{ctr} \\
 \hline
 \vdash \Gamma, \Delta \\
 \hline
 \vdash \Gamma, \Delta, \Delta \\
 \text{ctr}
 \end{array}
 \leftarrow
 \begin{array}{c}
 \begin{array}{c} \vdots P \\ \vdots Q \\ \hline \vdash \Gamma, F, F \end{array} \text{ctr} \quad \begin{array}{c} \vdots Q \\ \hline \vdash \Delta, \bar{F}, \bar{F} \end{array} \text{ctr} \\
 \hline
 \vdash \Gamma, F \quad \vdash \Delta, \bar{F} \\
 \hline
 \vdash \Gamma, \Delta \\
 \text{cut}
 \end{array}
 \rightarrow
 \begin{array}{c}
 \begin{array}{c} \vdots P \\ \vdots Q \\ \hline \vdash \Gamma, F, F \end{array} \text{ctr} \quad \begin{array}{c} \vdots P \\ \vdots Q \\ \hline \vdash \Gamma, F, F \end{array} \text{ctr} \\
 \hline
 \vdash \Gamma, F \quad \vdash \Gamma, F \\
 \hline
 \vdash \Gamma, \Delta \\
 \text{cut}
 \end{array}$$

# LAFONT'S EXAMPLE & CO. (PROOFS AND TYPES, 1989)

$$\frac{\frac{\frac{\vdots P}{\vdash \Gamma} \text{wk}}{\vdash \Gamma, \Gamma} \text{ctr}}{\vdash \Gamma} = P' \quad \leftarrow \quad \frac{\frac{\frac{\vdots P}{\vdash \Gamma} \text{wk} \quad \frac{\frac{\vdots Q}{\vdash \Gamma} \text{wk}}{\vdash \Gamma, \bar{F}} \text{wk}}{\vdash \Gamma, F} \text{cut}}{\vdash \Gamma, \Gamma} \text{ctr}}{\vdash \Gamma} \quad \rightarrow \quad Q' = \frac{\frac{\frac{\vdots Q}{\vdash \Gamma} \text{wk}}{\vdash \Gamma, \Gamma} \text{ctr}}{\vdash \Gamma}$$

**Assumption 1.** *There is a denotational interpretation  $\llbracket - \rrbracket : \text{LK} \rightarrow X$  mapping LK derivations to denotations in some space  $X$ .*

**Assumption 2.** *The interpretation is such that  $\llbracket P' \rrbracket = \llbracket P \rrbracket$  and  $\llbracket Q' \rrbracket = \llbracket Q \rrbracket$ .*

**Fact.**  $\llbracket P' \rrbracket = \llbracket Q' \rrbracket$ , hence  $\llbracket P \rrbracket = \llbracket Q \rrbracket$ .

# LAFONT'S EXAMPLE & CO. (PROOFS AND TYPES, 1989)

$$\begin{array}{c}
 \vdots P \\
 \frac{\vdots P}{\vdash F} \text{wk} \\
 \frac{\frac{\vdots P}{\vdash F} \text{wk}}{\vdash F, G} \vee \\
 \vdash F \vee G \\
 = P'
 \end{array}
 \leftarrow
 \begin{array}{c}
 \vdots P \quad \quad \quad \vdots Q \\
 \frac{\vdots P}{\vdash F} \text{wk} \quad \frac{\vdots Q}{\vdash G} \text{wk} \\
 \frac{\frac{\vdots P}{\vdash F} \text{wk} \quad \frac{\vdots Q}{\vdash G} \text{wk}}{\vdash F, H \quad \vdash G, \bar{H}} \text{cut} \\
 \frac{\vdash F, G}{\vdash F \vee G} \vee
 \end{array}
 \rightarrow
 \begin{array}{c}
 \vdots Q \\
 \frac{\vdots Q}{\vdash G} \text{wk} \\
 \frac{\frac{\vdots Q}{\vdash G} \text{wk}}{\vdash F, G} \vee \\
 \vdash F \vee G \\
 = Q'
 \end{array}$$

**Assumption 1.** *There is a denotational interpretation  $\llbracket - \rrbracket : \text{LK} \rightarrow X$  mapping LK derivations to denotations in some space  $X$ .*

**Fact.**  $\llbracket P' \rrbracket = \llbracket Q' \rrbracket$ .

# LAFONT'S EXAMPLE & CO. (PROOFS AND TYPES, 1989)

$$\frac{\frac{\vdots P \quad \vdots Q}{\vdash \Gamma \quad \vdash \Delta} \text{mix}}{\vdash \Gamma, \Delta} \quad \xleftarrow{?} \quad \frac{\frac{\frac{\vdots P}{\vdash \Gamma} \text{wk} \quad \frac{\vdots Q}{\vdash \Delta} \text{wk}}{\vdash \Gamma, F} \quad \frac{\vdots Q}{\vdash \Delta, \bar{F}} \text{wk}}{\vdash \Gamma, \Delta} \text{cut} \quad \xrightarrow{?} \quad \frac{\frac{\frac{\vdots P}{\vdash \Gamma} \text{wk} \quad \frac{\vdots Q}{\vdash \Delta} \text{wk}}{\vdash \Gamma, \Delta} \text{wk} \quad \frac{\vdots Q}{\vdash \Gamma, \Delta} \text{wk}}{\vdash \Gamma, \Delta} \oplus$$

# PATHOLOGICAL CRITICAL PAIRS

$$\begin{array}{c}
 \begin{array}{c} \vdots P \\ \vdots Q \\ \hline \vdash \Gamma, \bar{F}, \bar{G} \quad \vdash \Delta_1, F \end{array} \text{cut} \quad \begin{array}{c} \vdots R \\ \hline \vdash \Delta_2, G \end{array} \\
 \hline
 \vdash \Gamma, \Delta_1, \bar{G} \quad \vdash \Delta_2, G \\
 \hline
 \vdash \Gamma, \Delta_1, \Delta_2 \text{cut}
 \end{array}
 \leftarrow
 \begin{array}{c}
 \begin{array}{c} \vdots P \\ \hline \vdash \Gamma, \bar{F}, \bar{G} \end{array} \vee \quad \begin{array}{c} \vdots Q \quad \vdots R \\ \hline \vdash \Delta_1, F \quad \vdash \Delta_2, G \end{array} \wedge \\
 \hline
 \vdash \Gamma, \bar{F} \vee \bar{G} \quad \vdash \Delta_1, \Delta_2, F \wedge G \\
 \hline
 \vdash \Gamma, \Delta_1, \Delta_2 \text{cut}
 \end{array}
 \rightarrow
 \begin{array}{c}
 \begin{array}{c} \vdots P \quad \vdots R \\ \hline \vdash \Gamma, \bar{F}, \bar{G} \quad \vdash \Delta_2, G \end{array} \text{cut} \quad \begin{array}{c} \vdots Q \\ \hline \vdash \Delta_1, F \end{array} \\
 \hline
 \vdash \Gamma, \bar{F}, \Delta_2 \quad \vdash \Delta_1, F \\
 \hline
 \vdash \Gamma, \Delta_1, \Delta_2 \text{cut}
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{c} \vdots P \\ \hline \vdash \Gamma \end{array} \text{wk} \\
 \hline
 \vdash \Gamma, \Delta \text{wk}
 \end{array}
 \leftarrow
 \begin{array}{c}
 \begin{array}{c} \vdots P \quad \vdots Q \\ \hline \vdash \Gamma \quad \vdash \Delta \end{array} \text{wk} \\
 \hline
 \vdash \Gamma, F \quad \vdash \Delta, \bar{F} \\
 \hline
 \vdash \Gamma, \Delta \text{cut}
 \end{array}
 \rightarrow
 \begin{array}{c}
 \begin{array}{c} \vdots Q \\ \hline \vdash \Delta \end{array} \text{wk} \\
 \hline
 \vdash \Gamma, \Delta \text{wk}
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{c} \vdots P \quad \vdots Q \\ \hline \vdash \Gamma, F, F \quad \vdash \Delta, \bar{F}, \bar{F} \end{array} \text{ctr} \quad \begin{array}{c} \vdots Q \\ \hline \vdash \Delta, \bar{F}, \bar{F} \end{array} \text{ctr} \\
 \hline
 \vdash \Gamma, \Delta \quad \vdash \Delta, \bar{F} \\
 \hline
 \vdash \Gamma, \Delta, \Delta \\
 \hline
 \vdash \Gamma, \Delta \text{ctr}
 \end{array}
 \leftarrow
 \begin{array}{c}
 \begin{array}{c} \vdots P \quad \vdots Q \\ \hline \vdash \Gamma, F, F \quad \vdash \Delta, \bar{F}, \bar{F} \end{array} \text{ctr} \\
 \hline
 \vdash \Gamma, F \quad \vdash \Delta, \bar{F} \\
 \hline
 \vdash \Gamma, \Delta \text{cut}
 \end{array}
 \rightarrow
 \begin{array}{c}
 \begin{array}{c} \vdots P \quad \vdots Q \\ \hline \vdash \Gamma, F, F \quad \vdash \Delta, \bar{F}, \bar{F} \end{array} \text{ctr} \\
 \hline
 \vdash \Gamma, F \quad \vdash \Gamma, \Delta \\
 \hline
 \vdash \Gamma, \Gamma \\
 \hline
 \vdash \Gamma, \Delta \text{ctr}
 \end{array}$$



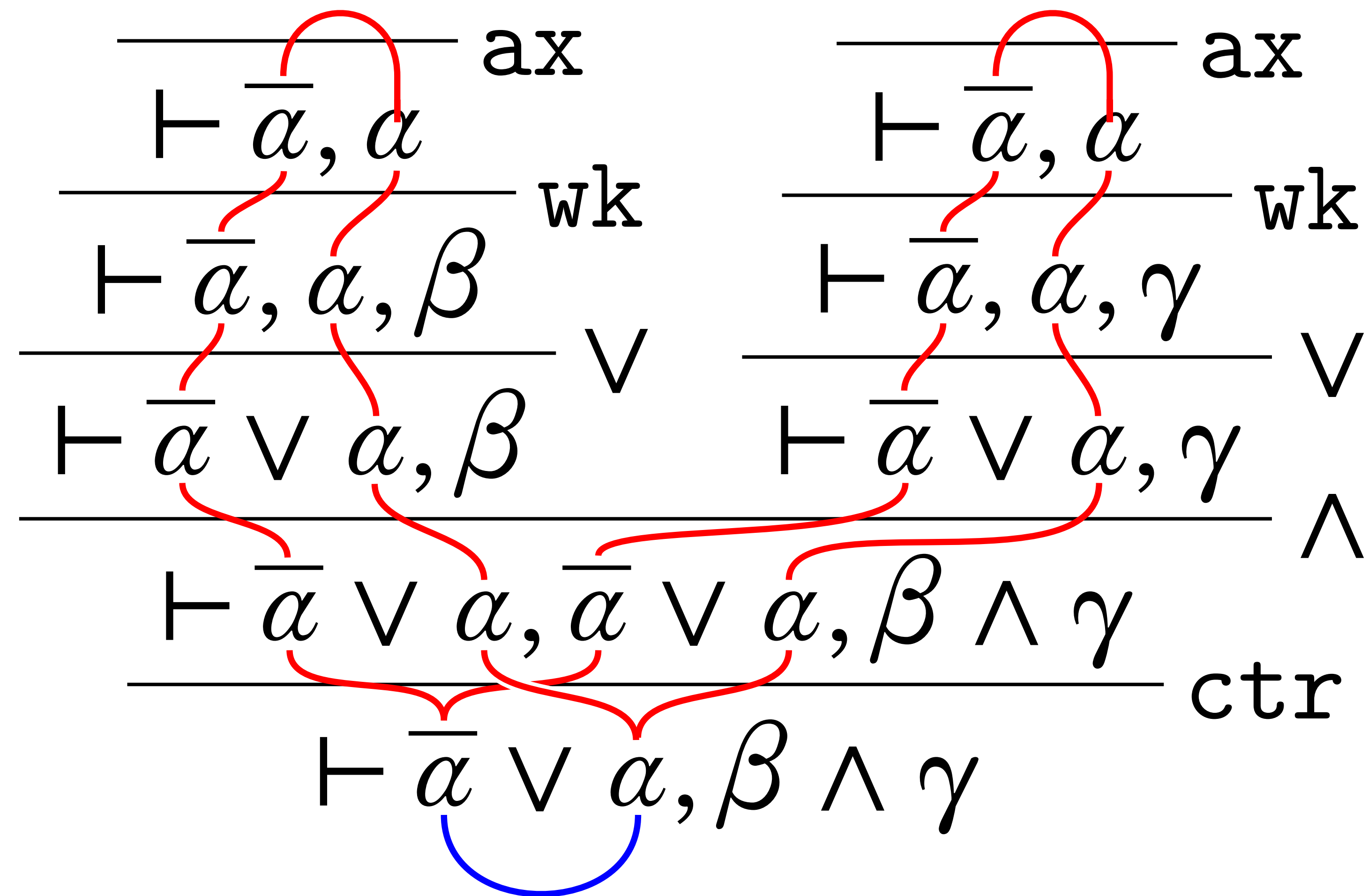
# IDEA #1 – TRACKING AXIOMS

Track existence, avoid counting.  
 (Andrews 1976, Lamarche & Straßburger 2004+)

$$\begin{array}{c}
 \frac{}{\vdash \bar{\alpha}, \alpha} \text{ ax} \quad \frac{}{\vdash \bar{\alpha}, \alpha} \text{ ax} \\
 \frac{\vdash \bar{\alpha}, \alpha}{\vdash \bar{\alpha}, \alpha, \beta} \text{ wk} \quad \frac{\vdash \bar{\alpha}, \alpha}{\vdash \bar{\alpha}, \alpha, \gamma} \text{ wk} \\
 \frac{\vdash \bar{\alpha}, \alpha, \beta}{\vdash \bar{\alpha} \vee \alpha, \beta} \vee \quad \frac{\vdash \bar{\alpha}, \alpha, \gamma}{\vdash \bar{\alpha} \vee \alpha, \gamma} \vee \\
 \frac{\vdash \bar{\alpha} \vee \alpha, \beta \quad \vdash \bar{\alpha} \vee \alpha, \gamma}{\vdash \bar{\alpha} \vee \alpha, \bar{\alpha} \vee \alpha, \beta \wedge \gamma} \wedge \\
 \frac{\vdash \bar{\alpha} \vee \alpha, \bar{\alpha} \vee \alpha, \beta \wedge \gamma}{\vdash \bar{\alpha} \vee \alpha, \beta \wedge \gamma} \text{ ctr}
 \end{array}$$

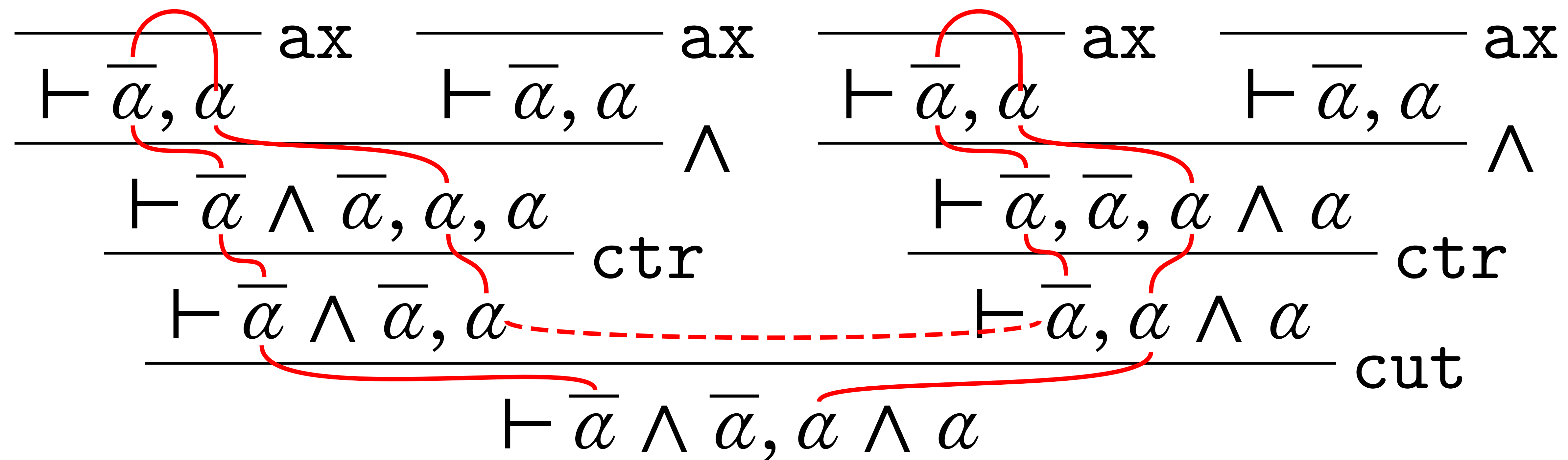
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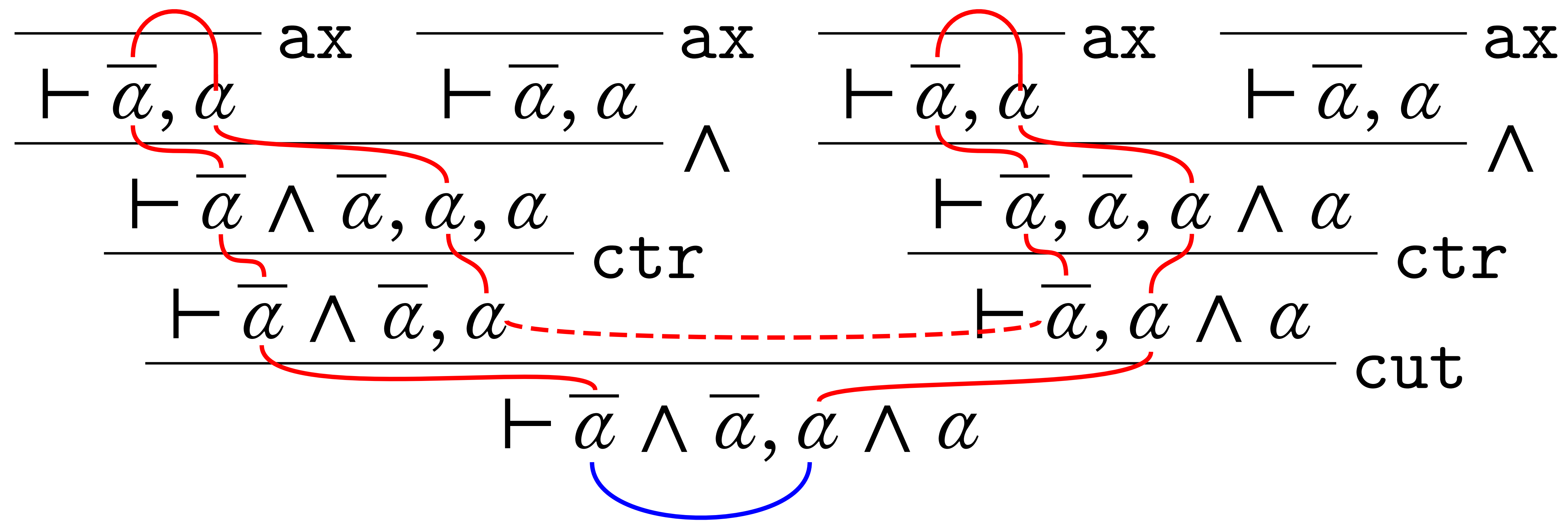
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 (Andrews 1976, Lamarche & Straßburger 2004+)



# IDEA #1 – TRACKING AXIOMS

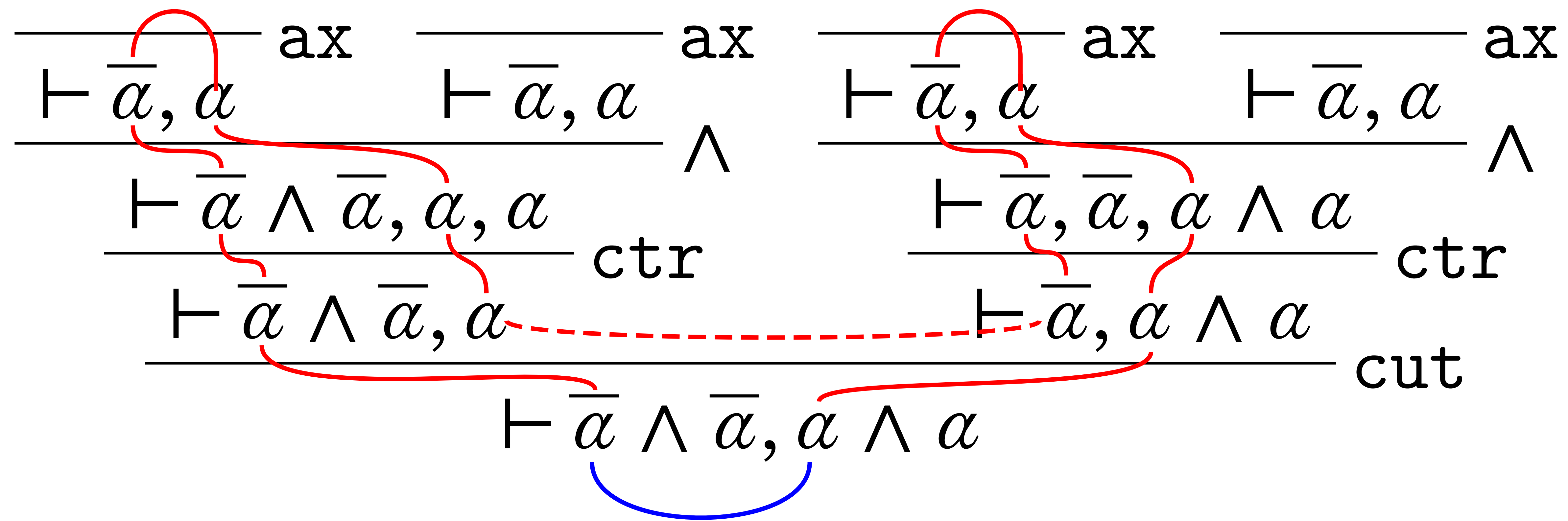
Track existence, avoid counting.  
 (Andrews 1976, Lamarche & Straßburger 2004+)



# IDEA #1 – TRACKING AXIOMS

Track existence, avoid counting.  
 (Andrews 1976, Lamarche & Straßburger 2004+)

**Theorem (Führmann & Pym 2004).** *Mating graphs decrease under cut-reduction.*



# IDEA #1 – TRACKING AXIOMS

Track existence, avoid counting.  
 (Andrews 1976, Lamarche & Straßburger 2004+)

Axioms may be preserved, deleted,  
 duplicated, but *never created*.

$$\begin{array}{c}
 \frac{}{\vdash \bar{F}, F} \text{ ax} \quad \frac{\vdots}{\vdash G, H, H} \text{ ctr} \quad \frac{\vdots}{\vdash \bar{H}, \bar{H}} \text{ ctr} \\
 \frac{\vdash \bar{F}, F \quad \vdash G, H}{\vdash \bar{F}, F \wedge G, H} \wedge \quad \frac{\vdash \bar{H}, \bar{H}}{\vdash \bar{H}} \text{ ctr} \\
 \frac{\vdash \bar{F}, F \wedge G, H \quad \vdash \bar{H}}{\vdash \bar{F}, F \wedge G} \text{ cut}
 \end{array}$$

# IDEA #1 – TRACKING AXIOMS

Track existence, avoid counting.  
 (Andrews 1976, Lamarche & Straßburger 2004+)

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$$\frac{\frac{\frac{}{\vdash \bar{F}, F} \text{ ax}}{\vdash \bar{F}, F} \quad \frac{\frac{\frac{\vdots}{\vdash G, H, H} \text{ ctr}}{\vdash G, H} \quad \frac{\frac{\frac{\vdots}{\vdash \bar{H}, \bar{H}} \text{ ctr}}{\vdash \bar{H}} \text{ cut}}{\vdash G} \wedge}{\vdash \bar{F}, F \wedge G} \wedge}$$





# MATINGS ARE NOT INVARIANT UNDER DUPLICATION!

$$\frac{
 \frac{
 \frac{
 \vdots P \quad \vdots Q
 }{\vdash F \quad \vdash G, F} \wedge
 \quad
 \vdots R
 }{\vdash F \wedge G, F \wedge G} \wedge
 }{\vdash F \wedge G} \text{ctr}
 \quad
 \frac{
 \frac{
 \frac{}{\vdash \bar{F}, F} \text{ax}
 }{\vdash \bar{F}, \bar{G}, F} \text{wk}
 \quad
 \frac{
 \frac{}{\vdash \bar{G}, G} \text{ax}
 }{\vdash \bar{F}, \bar{G}, G} \text{wk}
 }{\vdash \bar{F} \vee \bar{G}, F \quad \vdash \bar{F} \vee \bar{G}, G} \vee
 }{\vdash \bar{F} \vee \bar{G}, \bar{F} \vee \bar{G}, F \wedge G} \wedge
 }{\vdash \bar{F} \vee \bar{G}, F \wedge G} \text{ctr}
 }{\vdash \bar{F} \vee \bar{G}, F \wedge G} \text{cut}
 }{\vdash F \wedge G}$$

# MATINGS ARE NOT INVARIANT UNDER DUPLICATION!

$$\begin{array}{c}
 \begin{array}{c}
 \vdots P \\
 \vdots Q \\
 \vdots R
 \end{array} \\
 \frac{\frac{\frac{\vdots P}{\vdash F} \quad \frac{\frac{\vdots Q}{\vdash G, F}}{\vdash F \wedge G, F} \wedge \quad \vdots R}{\vdash F \wedge G, F \wedge G} \wedge}{\vdash F \wedge G} \text{ctr}
 \end{array}
 \quad
 \begin{array}{c}
 \frac{\frac{\frac{\text{---}}{\vdash \bar{F}, F} \text{ax}}{\vdash \bar{F}, \bar{G}, F} \text{wk}}{\vdash \bar{F} \vee \bar{G}, F} \vee \quad \frac{\frac{\frac{\text{---}}{\vdash \bar{G}, G} \text{ax}}{\vdash \bar{F}, \bar{G}, G} \text{wk}}{\vdash \bar{F} \vee \bar{G}, G} \vee}{\vdash \bar{F} \vee \bar{G}, F \wedge G} \wedge}{\vdash \bar{F} \vee \bar{G}, \bar{F} \vee \bar{G}, F \wedge G} \text{ctr} \\
 \frac{\vdash \bar{F} \vee \bar{G}, F \wedge G}{\vdash F \wedge G} \text{cut}
 \end{array}
 \end{array}$$

# MATINGS ARE NOT INVARIANT UNDER DUPLICATION!

$$\begin{array}{c}
 \begin{array}{c}
 \vdots P \\
 \vdots Q \\
 \vdots R
 \end{array} \\
 \frac{\frac{\frac{\vdots P}{\vdash F} \quad \frac{\vdots Q}{\vdash G, F}}{\vdash F \wedge G, F} \wedge \quad \vdots R}{\vdash F \wedge G, F \wedge G} \wedge \\
 \frac{\vdash F \wedge G, F \wedge G}{\vdash F \wedge G} \text{ctr} \\
 \hline
 \vdash F \wedge G
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{\frac{\frac{}{\vdash \bar{F}, F} \text{ax}}{\vdash \bar{F}, F} \text{wk}}{\vdash \bar{F}, \bar{G}, F} \vee \quad \frac{\frac{\frac{}{\vdash \bar{G}, G} \text{ax}}{\vdash \bar{G}, G} \text{wk}}{\vdash \bar{F}, \bar{G}, G} \vee}}{\vdash \bar{F} \vee \bar{G}, F} \vee \quad \frac{\vdash \bar{F} \vee \bar{G}, G}{\vdash \bar{F} \vee \bar{G}, G} \vee \\
 \frac{\vdash \bar{F} \vee \bar{G}, F \quad \vdash \bar{F} \vee \bar{G}, G}{\vdash \bar{F} \vee \bar{G}, F \wedge G} \wedge \\
 \frac{\vdash \bar{F} \vee \bar{G}, \bar{F} \vee \bar{G}, F \wedge G}{\vdash \bar{F} \vee \bar{G}, F \wedge G} \text{ctr} \\
 \hline
 \vdash \bar{F} \vee \bar{G}, F \wedge G \\
 \hline
 \vdash F \wedge G \quad \text{cut}
 \end{array}
 \end{array}$$

# MATINGS ARE NOT INVARIANT UNDER DUPLICATION!

$$\begin{array}{c}
 \begin{array}{c}
 \vdots P \\
 \vdots Q \\
 \vdots R
 \end{array} \\
 \frac{\frac{\frac{\vdots P}{\vdash F} \quad \frac{\frac{\vdots Q}{\vdash G, F}}{\vdash G, F} \wedge}{\vdash F \wedge G, F} \quad \frac{\vdots R}{\vdash G}}{\vdash F \wedge G, F \wedge G} \wedge}{\vdash F \wedge G} \text{ctr}
 \end{array}
 \quad
 \begin{array}{c}
 \frac{\frac{\frac{}{\vdash \bar{F}, F} \text{ax}}{\vdash \bar{F}, F} \text{wk}}{\vdash \bar{F}, \bar{G}, F} \vee}{\vdash \bar{F} \vee \bar{G}, F} \vee}{\vdash \bar{F} \vee \bar{G}, \bar{F} \vee \bar{G}, F \wedge G} \wedge}{\vdash \bar{F} \vee \bar{G}, \bar{F} \vee \bar{G}, F \wedge G} \text{ctr}} \\
 \frac{\frac{\frac{\frac{}{\vdash \bar{G}, G} \text{ax}}{\vdash \bar{G}, G} \text{wk}}{\vdash \bar{F}, \bar{G}, G} \vee}{\vdash \bar{F} \vee \bar{G}, G} \vee}{\vdash \bar{F} \vee \bar{G}, \bar{F} \vee \bar{G}, F \wedge G} \wedge}{\vdash \bar{F} \vee \bar{G}, \bar{F} \vee \bar{G}, F \wedge G} \text{ctr}} \\
 \frac{\frac{\vdash \bar{F} \vee \bar{G}, \bar{F} \vee \bar{G}, F \wedge G}{\vdash \bar{F} \vee \bar{G}, F \wedge G} \text{ctr}}{\vdash \bar{F} \vee \bar{G}, F \wedge G} \text{cut}
 \end{array}
 \end{array}$$

The diagram illustrates a derivation in a sequent calculus. On the left, a derivation for  $\vdash F \wedge G$  is shown. It starts with three vertical ellipses labeled  $P$ ,  $Q$ , and  $R$ .  $P$  leads to  $\vdash F$ .  $Q$  leads to  $\vdash G, F$ . These are combined via an  $\wedge$  rule to get  $\vdash F \wedge G, F$ . Then,  $R$  leads to  $\vdash G$ , and another  $\wedge$  rule yields  $\vdash F \wedge G, F \wedge G$ . Finally, a contraction rule (ctr) gives  $\vdash F \wedge G$ . A red solid line loops around the  $\vdash G, F$  and  $\vdash G$  parts, and a red dashed line loops around the  $\vdash F \wedge G, F \wedge G$  and  $\vdash F \wedge G$  parts. On the right, a derivation for  $\vdash \bar{F} \vee \bar{G}, F \wedge G$  is shown. It starts with two vertical ellipses labeled  $\bar{F}$  and  $\bar{G}$ .  $\bar{F}$  leads to  $\vdash \bar{F}, F$  via an axiom (ax) rule, which is then weakened (wk) to  $\vdash \bar{F}, \bar{G}, F$ .  $\bar{G}$  leads to  $\vdash \bar{G}, G$  via an axiom (ax) rule, which is then weakened (wk) to  $\vdash \bar{F}, \bar{G}, G$ . These are combined via a disjunction ( $\vee$ ) rule to get  $\vdash \bar{F} \vee \bar{G}, F$  and  $\vdash \bar{F} \vee \bar{G}, G$ . These are then combined via an  $\wedge$  rule to get  $\vdash \bar{F} \vee \bar{G}, \bar{F} \vee \bar{G}, F \wedge G$ . Finally, a contraction rule (ctr) gives  $\vdash \bar{F} \vee \bar{G}, \bar{F} \vee \bar{G}, F \wedge G$ . Another contraction rule (ctr) gives  $\vdash \bar{F} \vee \bar{G}, F \wedge G$ . Finally, a cut rule (cut) combines the two main derivations to yield  $\vdash F \wedge G$ . Blue solid lines connect the  $\vdash \bar{F}, \bar{G}, F$  and  $\vdash \bar{F}, \bar{G}, G$  parts to the  $\vdash \bar{F} \vee \bar{G}, \bar{F} \vee \bar{G}, F \wedge G$  part. A red dashed line loops around the  $\vdash \bar{F} \vee \bar{G}, F \wedge G$  and  $\vdash F \wedge G$  parts.

# MATINGS ARE NOT INVARIANT UNDER DUPLICATION!

$$\begin{array}{c}
 \begin{array}{c}
 \vdots P \\
 \vdots Q \\
 \vdots R
 \end{array} \\
 \frac{\frac{\frac{\vdots P}{\vdash F} \quad \frac{\vdots Q}{\vdash G, F}}{\vdash F \wedge G, F} \wedge \quad \vdots R}{\vdash F \wedge G, F \wedge G} \wedge \\
 \frac{\vdash F \wedge G, F \wedge G}{\vdash F \wedge G} \text{ctr}
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{\frac{\frac{}{\vdash \bar{F}, F} \text{ax}}{\vdash \bar{F}, F} \text{wk}}{\vdash \bar{F}, \bar{G}, F} \vee \quad \frac{\frac{\frac{}{\vdash \bar{G}, G} \text{ax}}{\vdash \bar{G}, G} \text{wk}}{\vdash \bar{F}, \bar{G}, G} \vee}}{\vdash \bar{F} \vee \bar{G}, F} \vee \quad \frac{\vdash \bar{F} \vee \bar{G}, G}{\vdash \bar{F} \vee \bar{G}, G} \vee \\
 \frac{\vdash \bar{F} \vee \bar{G}, F \quad \vdash \bar{F} \vee \bar{G}, G}{\vdash \bar{F} \vee \bar{G}, F \wedge G} \wedge \\
 \frac{\vdash \bar{F} \vee \bar{G}, \bar{F} \vee \bar{G}, F \wedge G}{\vdash \bar{F} \vee \bar{G}, F \wedge G} \text{ctr} \\
 \frac{\vdash \bar{F} \vee \bar{G}, F \wedge G}{\vdash F \wedge G} \text{cut}
 \end{array}
 \end{array}$$

# MATINGS ARE NOT INVARIANT UNDER DUPLICATION!

$$\begin{array}{c}
 \begin{array}{c} \vdots P \\ \vdots Q \end{array} \\
 \frac{\frac{\frac{\vdots P}{\vdash F} \quad \frac{\vdots Q}{\vdash G, F}}{\vdash F \wedge G, F} \wedge \quad \frac{\vdots R}{\vdash G}}{\vdash F \wedge G, F \wedge G} \wedge \\
 \frac{\vdash F \wedge G, F \wedge G}{\vdash F \wedge G} \text{ctr}
 \end{array}
 \quad
 \begin{array}{c}
 \frac{\frac{\frac{}{\vdash \bar{F}, F} \text{ax}}{\vdash \bar{F}, F} \text{wk}}{\vdash \bar{F}, \bar{G}, F} \vee \quad \frac{\frac{\frac{}{\vdash \bar{G}, G} \text{ax}}{\vdash \bar{G}, G} \text{wk}}{\vdash \bar{F}, \bar{G}, G} \vee}}{\vdash \bar{F} \vee \bar{G}, F} \vee \quad \frac{\vdash \bar{F} \vee \bar{G}, G}{\vdash \bar{F} \vee \bar{G}, G} \vee \\
 \frac{\vdash \bar{F} \vee \bar{G}, F \quad \vdash \bar{F} \vee \bar{G}, G}{\vdash \bar{F} \vee \bar{G}, F \wedge G} \wedge \\
 \frac{\vdash \bar{F} \vee \bar{G}, \bar{F} \vee \bar{G}, F \wedge G}{\vdash \bar{F} \vee \bar{G}, F \wedge G} \text{ctr} \\
 \frac{\vdash \bar{F} \vee \bar{G}, F \wedge G}{\vdash \bar{F} \vee \bar{G}, F \wedge G} \text{cut}
 \end{array}
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{c} \vdots P \\ \vdots R \end{array} \\
 \frac{\frac{\frac{\vdots P}{\vdash F} \text{wk}}{\vdash F, F} \text{ctr} \quad \frac{\frac{\vdots R}{\vdash G} \text{wk}}{\vdash G, G} \text{ctr}}{\vdash F} \text{ctr} \quad \frac{\vdash G}{\vdash G} \text{ctr} \\
 \frac{\vdash F \quad \vdash G}{\vdash F \wedge G} \wedge
 \end{array}$$

## IDEA #2 – TARGET INVERTIBILITY

*Identity group:*

$$\frac{}{\vdash \Gamma, A, \bar{A}} \text{ax} \quad \frac{\vdash \Gamma, A \quad \vdash \Gamma, \bar{A}}{\vdash \Gamma} \text{cut}$$

*Structural group:*

$$\frac{\vdash \Gamma \quad \vdash \Gamma}{\vdash \Gamma} \text{sum}$$

*Logical group:*

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B} \vee \quad \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B} \wedge$$

## IDEA #2 – TARGET INVERTIBILITY

$$\frac{\frac{\frac{\vdash A, B, C \quad \vdash A, B, D}{\vdash A, B, C \wedge D} \wedge}{\vdash A \vee B, C \wedge D} \vee}{\vdash A \vee B, C \wedge D} \vee \quad \longleftrightarrow \quad \frac{\frac{\frac{\vdash A, B, C}{\vdash A \vee B, C} \vee \quad \frac{\vdash A, B, D}{\vdash A \vee B, D} \vee}{\vdash A \vee B, C \wedge D} \wedge}{\vdash A \vee B, C \wedge D} \wedge$$

$$\frac{\frac{\frac{\vdash A, C \quad \vdash A, D}{\vdash A, C \wedge D} \wedge \quad \frac{\vdash B, C \quad \vdash B, D}{\vdash B, C \wedge D} \wedge}{\vdash A \wedge B, C \wedge D} \wedge}{\vdash A \wedge B, C \wedge D} \wedge \quad \longleftrightarrow \quad \frac{\frac{\frac{\vdash A, C \quad \vdash B, C}{\vdash A \wedge B, C} \wedge \quad \frac{\vdash A, D \quad \vdash B, D}{\vdash A \wedge B, D} \wedge}{\vdash A \wedge B, C \wedge D} \wedge}{\vdash A \wedge B, C \wedge D} \wedge$$



## IDEA #2 – TARGET INVERTIBILITY

$$\frac{\frac{\frac{\vdash A, B, C \quad \vdash A, B, D}{\vdash A, B, C \wedge D} \wedge}{\vdash A \vee B, C \wedge D} \vee}{\vdash A \vee B, C \wedge D} \wedge \quad \longleftrightarrow \quad \frac{\frac{\frac{\vdash A, B, C}{\vdash A \vee B, C} \vee \quad \frac{\vdash A, B, D}{\vdash A \vee B, D} \vee}{\vdash A \vee B, C \wedge D} \wedge}{\vdash A \vee B, C \wedge D} \wedge$$

$$\frac{\frac{\frac{\vdash A, C \quad \vdash A, D}{\vdash A, C \wedge D} \wedge \quad \frac{\vdash B, C \quad \vdash B, D}{\vdash B, C \wedge D} \wedge}{\vdash A \wedge B, C \wedge D} \wedge}{\vdash A \wedge B, C \wedge D} \wedge \quad \longleftrightarrow \quad \frac{\frac{\frac{\vdash A, C \quad \vdash B, C}{\vdash A \wedge B, C} \wedge \quad \frac{\vdash A, D \quad \vdash B, D}{\vdash A \wedge B, D} \wedge}{\vdash A \wedge B, C \wedge D} \wedge}{\vdash A \wedge B, C \wedge D} \wedge$$

**Theorem.** *Mating graphs are invariant under arbitrary permutations of logical rules in the cut-free fragment of GS3.*

### IDEA #3 — ADD BRANCH ANNOTATIONS

$$\begin{array}{c}
 \frac{\frac{}{\vdash \bar{\alpha}, \alpha, \alpha} \text{ax}}{\vdash \bar{\alpha} \vee \alpha, \alpha} \vee \quad \frac{\frac{}{\vdash \bar{\alpha}, \alpha, \alpha} \text{ax}}{\vdash \bar{\alpha} \vee \alpha, \alpha} \vee \quad \frac{\frac{}{\vdash \bar{\alpha}, \alpha, \bar{\alpha}} \text{ax}}{\vdash \bar{\alpha} \vee \alpha, \bar{\alpha}} \vee \quad \frac{\frac{}{\vdash \bar{\alpha}, \alpha, \bar{\alpha}} \text{ax}}{\vdash \bar{\alpha} \vee \alpha, \bar{\alpha}} \vee \\
 \frac{\frac{}{\vdash \bar{\alpha} \vee \alpha, \alpha} \wedge \quad \frac{\frac{}{\vdash \bar{\alpha} \vee \alpha, \bar{\alpha}} \wedge}{\vdash (\bar{\alpha} \vee \alpha) \wedge (\bar{\alpha} \vee \alpha), \alpha} \wedge \quad \frac{\frac{}{\vdash \bar{\alpha} \vee \alpha, \bar{\alpha}} \wedge}{\vdash (\bar{\alpha} \vee \alpha) \wedge (\bar{\alpha} \vee \alpha), \bar{\alpha}} \wedge \\
 \frac{\frac{}{\vdash (\bar{\alpha} \vee \alpha) \wedge (\bar{\alpha} \vee \alpha), \alpha} \text{cut} \quad \frac{\frac{}{\vdash (\bar{\alpha} \vee \alpha) \wedge (\bar{\alpha} \vee \alpha), \bar{\alpha}} \text{cut}}{\vdash (\bar{\alpha} \vee \alpha) \wedge (\bar{\alpha} \vee \alpha)} \text{cut}
 \end{array}$$

The diagram illustrates a logical derivation process. It starts with four axioms (ax) at the top, each leading to a disjunction (∨) in the second row. These are then combined using conjunction (∧) in the third row. Finally, a cut rule is applied in the fourth row to derive the final result,  $\vdash (\bar{\alpha} \vee \alpha) \wedge (\bar{\alpha} \vee \alpha)$ . Red arrows and a dashed red line highlight the flow of information from the axioms to the final result. A blue arrow highlights the final result.

### IDEA #3 – ADD BRANCH ANNOTATIONS

$$\begin{array}{c}
 \frac{\frac{\frac{}{\vdash \bar{\alpha}, \alpha, \alpha} \text{ax}}{\vdash \bar{\alpha} \vee \alpha, \alpha} \vee}{\vdash (\bar{\alpha} \vee \alpha) \wedge (\bar{\alpha} \vee \alpha), \alpha} \wedge}{\vdash (\bar{\alpha} \vee \alpha) \wedge (\bar{\alpha} \vee \alpha)} \text{cut} \quad
 \frac{\frac{\frac{}{\vdash \bar{\alpha}, \alpha, \alpha} \text{ax}}{\vdash \bar{\alpha} \vee \alpha, \alpha} \vee}{\vdash (\bar{\alpha} \vee \alpha) \wedge (\bar{\alpha} \vee \alpha), \bar{\alpha}} \wedge}{\vdash (\bar{\alpha} \vee \alpha) \wedge (\bar{\alpha} \vee \alpha)} \text{cut}
 \end{array}$$

$$\frac{\frac{\frac{}{\vdash \bar{\alpha}, \alpha, \alpha} \text{ax}}{\vdash \bar{\alpha} \vee \alpha, \alpha} \vee}{\vdash \bar{\alpha} \vee \alpha} \text{cut} \quad \frac{\frac{\frac{}{\vdash \bar{\alpha}, \alpha, \bar{\alpha}} \text{ax}}{\vdash \bar{\alpha} \vee \alpha, \bar{\alpha}} \vee}{\vdash \bar{\alpha} \vee \alpha} \text{cut}}{\vdash (\bar{\alpha} \vee \alpha) \wedge (\bar{\alpha} \vee \alpha)} \wedge$$

# IDEA #3 – ADD BRANCH ANNOTATIONS

(1)	(2)	(3)	(4)
$\frac{\frac{}{\vdash \bar{\alpha}, \alpha, \alpha} \text{ax}}{\vdash \bar{\alpha} \vee \alpha, \alpha} \vee$	$\frac{}{\vdash \bar{\alpha}, \alpha, \alpha} \text{ax}$	$\frac{}{\vdash \bar{\alpha}, \alpha, \bar{\alpha}} \text{ax}$	$\frac{}{\vdash \bar{\alpha}, \alpha, \bar{\alpha}} \text{ax}$
$\frac{}{\vdash \bar{\alpha} \vee \alpha, \alpha} \vee$	$\frac{}{\vdash \bar{\alpha} \vee \alpha, \alpha} \vee$	$\frac{}{\vdash \bar{\alpha} \vee \alpha, \bar{\alpha}} \vee$	$\frac{}{\vdash \bar{\alpha} \vee \alpha, \bar{\alpha}} \vee$
$\frac{}{\vdash (\bar{\alpha} \vee \alpha) \wedge (\bar{\alpha} \vee \alpha), \alpha} \wedge$		$\frac{}{\vdash (\bar{\alpha} \vee \alpha) \wedge (\bar{\alpha} \vee \alpha), \bar{\alpha}} \wedge$	
$\frac{}{\vdash (\bar{\alpha} \vee \alpha) \wedge (\bar{\alpha} \vee \alpha)} \text{cut}$			

(1)	(3)	(2)	(4)
$\frac{}{\vdash \bar{\alpha}, \alpha, \alpha} \text{ax}$	$\frac{}{\vdash \bar{\alpha}, \alpha, \bar{\alpha}} \text{ax}$	$\frac{}{\vdash \bar{\alpha}, \alpha, \alpha} \text{ax}$	$\frac{}{\vdash \bar{\alpha}, \alpha, \bar{\alpha}} \text{ax}$
$\frac{}{\vdash \bar{\alpha} \vee \alpha, \alpha} \vee$	$\frac{}{\vdash \bar{\alpha} \vee \alpha, \bar{\alpha}} \vee$	$\frac{}{\vdash \bar{\alpha} \vee \alpha, \alpha} \vee$	$\frac{}{\vdash \bar{\alpha} \vee \alpha, \bar{\alpha}} \vee$
$\frac{}{\vdash \bar{\alpha} \vee \alpha} \text{cut}$		$\frac{}{\vdash \bar{\alpha} \vee \alpha} \text{cut}$	
$\frac{}{\vdash \bar{\alpha} \vee \alpha} \wedge$		$\frac{}{\vdash \bar{\alpha} \vee \alpha} \wedge$	
$\vdash (\bar{\alpha} \vee \alpha) \wedge (\bar{\alpha} \vee \alpha)$			

# NAMED FORMULAS & SEQUENTS

$$A, B ::= \alpha_x \mid \bar{\alpha}_x \mid A \vee B \mid A \wedge B \quad (\alpha \in \mathcal{A}, x \in \mathcal{N})$$

$$\overline{(\alpha_x)} = \bar{\alpha}_x \quad \overline{(\bar{\alpha}_x)} = \alpha_x \quad \overline{A \vee B} = \bar{A} \wedge \bar{B} \quad \overline{A \wedge B} = \bar{A} \vee \bar{B} \quad (\textit{negation})$$

$$|\alpha_x| = \alpha \quad |\bar{\alpha}_x| = \bar{\alpha} \quad |A \vee B| = |A| \wedge |B| \quad |A \wedge B| = |A| \vee |B| \quad (\textit{underlying formula})$$

$$A \equiv B \iff |A| = |B| \quad (\textit{congruence})$$

$$\vdash A_1, \dots, A_n \quad (\textit{sequent})$$

**Sharing-free** formula, sequent, set of formulas: *each name occurs at most once.*

# NAMED DERIVATIONS

*Identity group:*

$$\frac{}{\vdash \Gamma, A, \bar{B}} \text{ax}_{\{A, \bar{B}\}} \quad (A \equiv B) \quad \frac{\vdash \Gamma, A \quad \vdash \Gamma, \bar{A}}{\vdash \Gamma} \text{cut}$$

*Structural group:*

$$\frac{\vdash \Gamma \quad \vdash \Gamma}{\vdash \Gamma} \sqcup$$

*Logical group:*

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B} \vee \quad \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B} \wedge$$

## BRANCH-LABELED MATINGS

Pairs  $G (H, K, \dots) = \langle V_G, \prec_G \rangle$

where  $V_G \subseteq \mathcal{N}$  is a set of names (*vertices*)

$\prec_G \subseteq \binom{V_G}{2} \times \mathcal{P}(V_G)$  is a relation between *unordered pairs* of vertices (*edges*) and arbitrary sets of vertices (*branch labels*),

such that  $e \prec_G X \implies e \subseteq X$

*Operations:*

$$G \sqcup H = \langle V_G \cup V_H, \prec_G \cup \prec_H \rangle$$

$$G \upharpoonright_X = \langle V_G \cap X, \{(e, Y) \in \prec_G \mid Y \subseteq X\} \rangle$$

*Subgraph relation:*

$$G \sqsubseteq H \iff V_G \subseteq V_H \wedge \prec_G \subseteq \prec_H$$

# BRANCH-SENSITIVE COMPOSITION

*Composition:*

$$G \odot_I H = \langle (V_G \cup V_H) \setminus I, \prec \rangle \quad (I \subseteq \mathcal{N})$$

where for all  $x \neq y \in (V_G \cup V_H) \setminus I$

$$xy \prec Z$$

if and only if there is a *complete alternating Z-labeled path*

$$z_1, \dots, z_n$$

between  $G$  and  $H$  through the interface  $I$ ,

such that  $z_1 = x$  and  $z_n = y$ .



# BRANCH-LABELED ALTERNATING PATH

*Labeling modulo interfaces:*

$$e \prec_G^I X \iff \exists Y. e \prec_G Y \wedge X = Y \setminus I$$

*Alternating X-labeled path between G and H through the interface I:*

a finite sequence  $x_1, \dots, x_n \in V_G \cup V_H$  of pairwise distinct vertices, such that

- (i)  $x_i \in I$  for all  $1 < i < n$  (internal vertices lie in the interface  $I$ )
- (ii) either  $e_i \prec_G^I X$  for all odd  $1 \leq i < n$  and  $e_i \prec_H^I X$  for all even  $1 \leq i < n$ ,  
or  $e_i \prec_H^I X$  for all odd  $1 \leq i < n$  and  $e_i \prec_G^I X$  for all even  $1 \leq i < n$ ,

where  $e_i = x_i x_{i+1}$ .

*Complete* if  $x_1, x_n \notin I$  (i.e. the endpoints lie outside the interface).

## THE INTERPRETATION

$$\left[ \frac{}{\vdash \Gamma, \alpha_x, \bar{\alpha}_y} \text{ax}_{\{\alpha_x, \bar{\alpha}_y\}} \right] = \langle \text{names}(\Gamma, \alpha_x, \bar{\alpha}_y), \{xy \prec \text{names}(\Gamma, \alpha_x, \bar{\alpha}_y)\} \rangle \quad (\Gamma \text{ atomic})$$

$$\left[ \frac{\begin{array}{c} \vdots P \\ \vdots Q \end{array}}{\vdash \Gamma, A \quad \vdash \Gamma, \bar{A}} \text{cut} \right] = \llbracket P \rrbracket \odot_A \llbracket Q \rrbracket$$

$$\left[ \frac{\begin{array}{c} \vdots P_1 \\ \vdots P_n \end{array}}{\vdash \Gamma_1 \quad \dots \quad \vdash \Gamma_n} \text{rule} \right] = \llbracket P_1 \rrbracket \sqcup \dots \sqcup \llbracket P_n \rrbracket \quad (\text{rule} \in \{\sqcup, \vee, \wedge\})$$

# MAIN RESULTS

**Theorem (Soundness).** *Branch-annotated mating graphs are invariant under arbitrary permutations of logical rules in the full calculus.*

*Proof sketch*

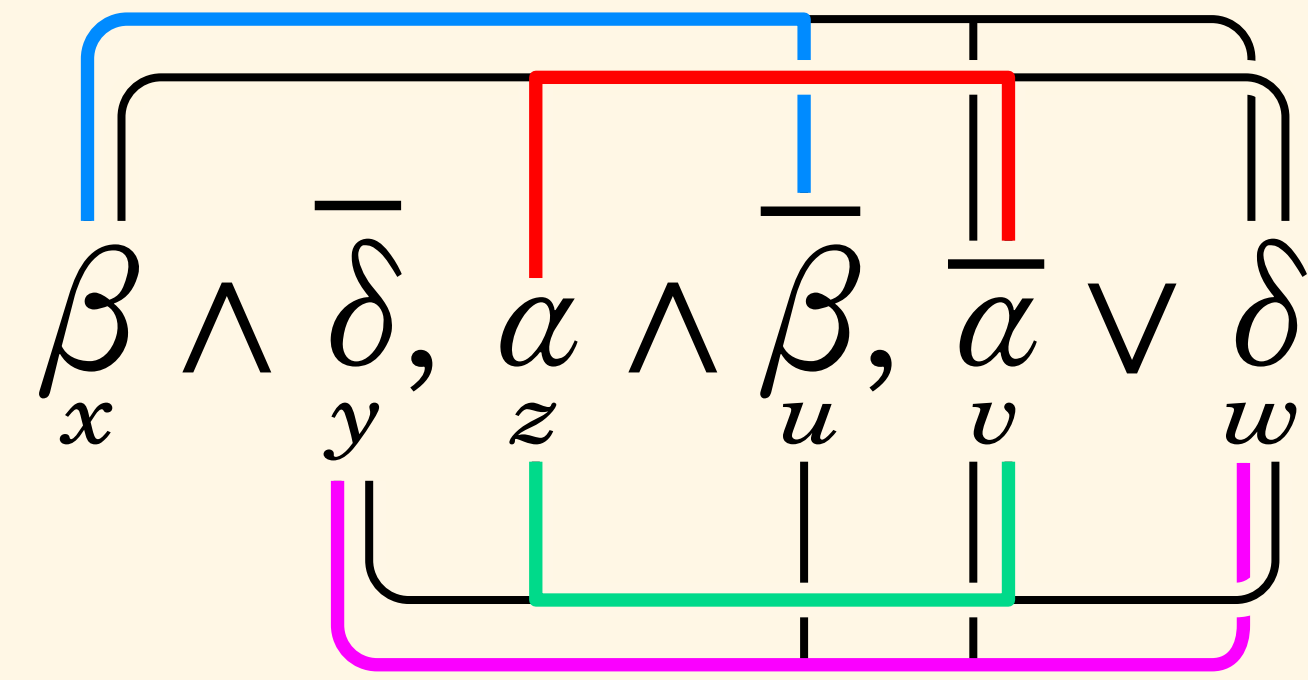
1. Use compositionality.
2. Trivial for logical, union rule commutations.
3. For commutations involving cuts, reason about alternating paths.  
The only complex case is the conjunction-cut one.

# MAIN RESULTS

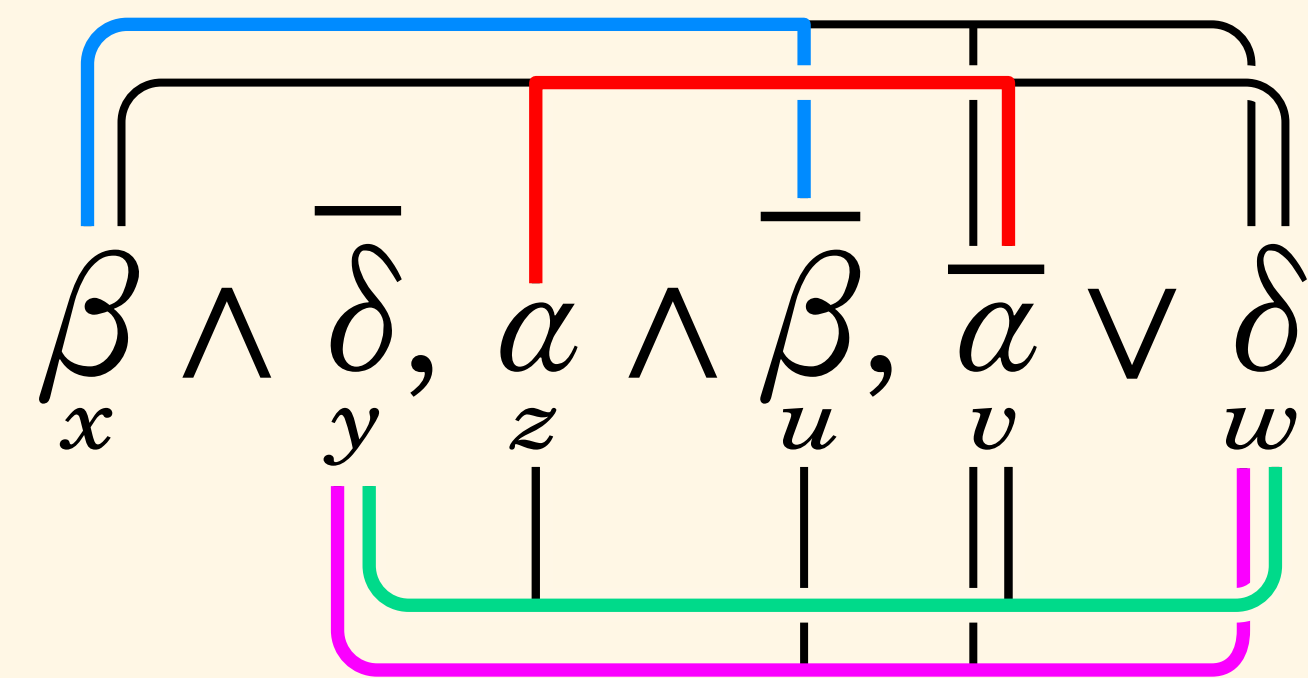
**Theorem (Cut-elimination).** *For any GS4 derivation there is a cut-free GS4 derivation with the same conclusion and denotation.*

*Proof sketch*

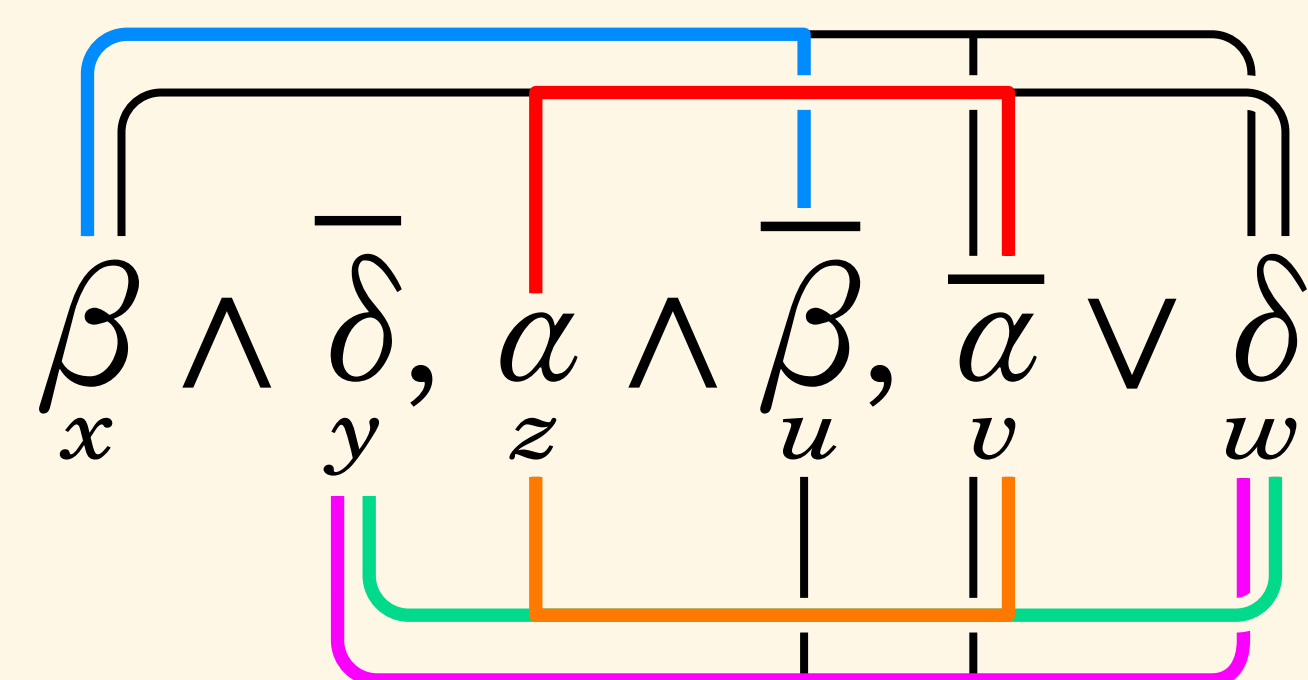
1. Upon finding a cut, normalize recursively its sub-derivations.
2. Use logical rule permutations to reduce the *conclusion* of the cut to an atomic clause.
3. Compute the denotation and reconstruct the resulting derivation.



$$\frac{\beta_x, \alpha_z, \bar{a}_v, \delta_w \quad \beta_x, \beta_u, \bar{a}_v, \delta_w \quad \bar{\delta}_y, \alpha_z, \bar{a}_v, \delta_w \quad \bar{\delta}_y, \beta_u, \bar{a}_v, \delta_w}{\beta_x \wedge \bar{\delta}_y, \alpha_z \wedge \beta_u, \bar{a}_v \vee \delta_w}$$



$$\frac{\beta_x, \alpha_z, \bar{a}_v, \delta_w \quad \beta_x, \beta_u, \bar{a}_v, \delta_w \quad \bar{\delta}_y, \alpha_z, \bar{a}_v, \delta_w \quad \bar{\delta}_y, \beta_u, \bar{a}_v, \delta_w}{\beta_x \wedge \bar{\delta}_y, \alpha_z \wedge \beta_u, \bar{a}_v \vee \delta_w}$$



$$\frac{\beta_x, \alpha_z, \bar{a}_v, \delta_w \quad \beta_x, \beta_u, \bar{a}_v, \delta_w \quad \bar{\delta}_y, \alpha_z, \bar{a}_v, \delta_w \quad \bar{\delta}_y, \beta_u, \bar{a}_v, \delta_w}{\beta_x \wedge \bar{\delta}_y, \alpha_z \wedge \beta_u, \bar{a}_v \vee \delta_w}$$

# MAIN RESULTS

**Theorem (Sequentialization).** *A branch-annotated mating  $G$  is correct w.r.t. some sequent  $\vdash \Gamma$  if and only if there is a GS4 derivation  $P \vdash \Gamma$  whose denotation is  $G$*

*Moreover, the size of  $P$  is polynomially bounded by the size of  $G$ .*

# MAIN RESULTS

**Theorem (Sequentialization).** *A branch-annotated mating  $G$  is correct w.r.t. some sequent  $\vdash \Gamma$  if and only if there is a GS4 derivation  $P \vdash \Gamma$  whose denotation is  $G$*

*Moreover, the size of  $P$  is polynomially bounded by the size of  $G$ .*

Rule permutations are identities

Efficient invertibility

Admissibility of rules by algebraic reasoning

# SHORTCOMINGS

- Cuts do not commute!
- No local cut-reduction procedure.
- Has the original problem been solved? Unclear.
  - No predicate calculus.



**THANK YOU**