NON-TRIVIAL INVARIANTS OF RULE PERMUTATIONS AND CUT-ELIMINATION IN CLASSICAL SEQUENT CALCULUS

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GENTZEN'S CALCULUS LK (PROPOSITIONAL FRAGMENT)

$$F, G ::= \alpha \mid \neg F \mid F \to G \mid F \vee G \mid F \wedge G \qquad (\alpha \in \mathcal{A})$$

Identity group:

$$\frac{}{F \vdash F}\,\texttt{ax} \qquad \frac{\Gamma \vdash \Delta, F \quad F, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\texttt{cut}$$

Structural group:

$$\frac{\Gamma, F, G, \Gamma' \vdash \Delta}{\Gamma, G, F, \Gamma' \vdash \Delta}\,\texttt{xch}\vdash \qquad \frac{\Gamma \vdash \Delta}{F, \Gamma \vdash \Delta}\,\texttt{wk}\vdash \qquad \frac{F, F, \Gamma \vdash \Delta}{F, \Gamma \vdash \Delta}\,\texttt{ctr}\vdash$$

$$\frac{\Gamma \vdash \Delta, F, G\Delta'}{\Gamma \vdash \Delta, G, F, \Delta'}\vdash\texttt{xch} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, F}\vdash\texttt{wk} \qquad \frac{\Gamma \vdash \Delta, F, F}{\Gamma \vdash \Delta, F, F}\vdash\texttt{ctr}$$

Additive group (context sharing):

$$\frac{F, \Gamma \vdash \Delta}{F \wedge G, \Gamma \vdash \Delta}\wedge_{\texttt{l}}\vdash \qquad \frac{\Gamma \vdash \Delta, F}{\Gamma \vdash \Delta, F \vee G}\vdash\vee_{\texttt{l}}$$

$$\frac{G, \Gamma \vdash \Delta}{F \wedge G, \Gamma \vdash \Delta}\wedge_{\texttt{r}}\vdash \qquad \frac{\Gamma \vdash \Delta, G}{\Gamma \vdash \Delta, F \vee G}\vdash\vee_{\texttt{r}}$$

$$\frac{\Gamma \vdash \Delta, F \quad \Gamma \vdash \Delta, G}{\Gamma \vdash \Delta, F \wedge G}\vdash\wedge \qquad \frac{F, \Gamma \vdash \Delta \quad G, \Gamma \vdash \Delta}{F \vee G, \Gamma \vdash \Delta}\vee\vdash$$

Multiplicative group (context splitting):

$$\frac{\Gamma \vdash \Delta, F}{\neg F, \Gamma \vdash \Delta}\neg\vdash \qquad \frac{\Gamma \vdash \Delta, F \quad G, \Gamma' \vdash \Delta'}{F \to G, \Gamma, \Gamma' \vdash \Delta, \Delta'}\to\vdash$$

$$\frac{F, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg F}\vdash\neg \qquad \frac{F, \Gamma \vdash \Delta, G}{\Gamma \vdash \Delta, F \to G}\vdash\to$$

THE CALCULUS LK (PROPOSITIONAL FRAGMENT, ONE SIDED)

$$F, G ::= \alpha \mid \overline{\alpha} \mid F \vee G \mid F \wedge G \qquad (\alpha \in \mathcal{A})$$

$$\overline{(\alpha)} = \overline{\alpha} \qquad \overline{(\overline{\alpha})} = \alpha \qquad \overline{F \vee G} = \overline{F} \wedge \overline{G} \qquad \overline{F \wedge G} = \overline{F} \vee \overline{G}$$

Identity group:

$$\frac{}{\vdash \overline{A}, A}\,\texttt{ax} \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, \overline{A}}{\vdash \Gamma, \Delta}\,\texttt{cut}$$

Structural group:

$$\frac{\vdash \Gamma}{\vdash \Gamma, A}\,\texttt{wk} \qquad \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\,\texttt{ctr}$$

Logical group:

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}\,\vee \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \wedge B}\,\wedge$$

CUT-ELIMINATION THEOREMS

Weak form. *The cut-rule is admissible.*

Strong form. *For every derivation* $P \vdash \Gamma$ *there is a cut-free derivation* $Q \vdash \Gamma$ *such that* $P \longrightarrow^{*} Q$.

How to design rewriting relations?

CUT-ELIMINATION THEOREMS

Weak form. *The cut-rule is admissible.*

Strong form. *For every derivation* $P \vdash \Gamma$ *there is a cut-free derivation* $Q \vdash \Gamma$ *such that* $P \longrightarrow^{*} Q$.

How to design rewriting relations?

a. Reduction: some measure of cut complexity should decrease strictly;

b. Invariants: some properties should be preserved
(at the very least, correctness and conclusions).

CUT-ELIMINATION THEOREMS

Weak form. *There is a rewriting relation that preserves correctness and conclusions and decreases the number of cuts.*

Strong form. *For every derivation* $P \vdash \Gamma$ *there is a cut-free derivation* $Q \vdash \Gamma$ *such that* $P \longrightarrow^* Q$.

How to design rewriting relations?

a. Reduction: some measure of cut complexity should decrease strictly;

b. Invariants: some properties should be preserved (at the very least, correctness and conclusions).

STUDYING INVARIANTS OF CUT-ELIMINATION

Denotational semantics:

Define functions $[\![-]\!] : \mathtt{LK} \to X$ mapping derivations to *denotations*, such that

$$P \longrightarrow Q \implies [\![P]\!] = [\![Q]\!]$$

STUDYING INVARIANTS OF CUT-ELIMINATION

Denotational semantics:

Define functions $[\![-]\!] : \mathtt{LK} \to X$ mapping derivations to *denotations*, such that

$$P \longrightarrow Q \implies [\![P]\!] = [\![Q]\!].$$

Coarsest: correctness and conclusions.

Finest: normal form (if unique).

Intermediate ones?

CUT-ELIMINATION — KEY CASES

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, P \\ \vdash \Gamma, \overline{F}, \overline{G}\end{array}}{\vdash \Gamma, \overline{F} \vee \overline{G}}\,\vee \qquad \dfrac{\begin{array}{c}\vdots\, Q \\ \vdash \Delta_1, F\end{array} \quad \begin{array}{c}\vdots\, R \\ \vdash \Delta_2, G\end{array}}{\vdash \Delta_1, \Delta_2, F \wedge G}\,\wedge}{\vdash \Gamma, \Delta_1, \Delta_2}\,\texttt{cut} \quad \longrightarrow \quad \dfrac{\dfrac{\begin{array}{c}\vdots\, P \\ \vdash \Gamma, \overline{F}, \overline{G}\end{array} \quad \begin{array}{c}\vdots\, Q \\ \vdash \Delta_1, F\end{array}}{\vdash \Gamma, \Delta_1, \overline{G}}\,\texttt{cut} \qquad \begin{array}{c}\vdots\, R \\ \vdash \Delta_2, G\end{array}}{\vdash \Gamma, \Delta_1, \Delta_2}\,\texttt{cut}$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, P \\ \vdash \Gamma, \overline{F}, \overline{G}\end{array}}{\vdash \Gamma, \overline{F} \vee \overline{G}}\,\vee \qquad \dfrac{\begin{array}{c}\vdots\, Q \\ \vdash \Delta_1, F\end{array} \quad \begin{array}{c}\vdots\, R \\ \vdash \Delta_2, G\end{array}}{\vdash \Delta_1, \Delta_2, F \wedge G}\,\wedge}{\vdash \Gamma, \Delta_1, \Delta_2}\,\texttt{cut} \quad \longrightarrow \quad \dfrac{\dfrac{\begin{array}{c}\vdots\, P \\ \vdash \Gamma, \overline{F}, \overline{G}\end{array} \quad \begin{array}{c}\vdots\, R \\ \vdash \Delta_2, G\end{array}}{\vdash \Gamma, \overline{F}, \Delta_2}\,\texttt{cut} \qquad \begin{array}{c}\vdots\, Q \\ \vdash \Delta_1, F\end{array}}{\vdash \Gamma, \Delta_1, \Delta_2}\,\texttt{cut}$$

CUT-ELIMINATION — STRUCTURAL CASES

$$\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma}}{\vdash \Gamma, F}\,\texttt{wk} \qquad \overset{\vdots\, Q}{\vdash \Delta, \overline{F}}}{\vdash \Gamma, \Delta}\,\texttt{cut} \quad \longrightarrow \quad \overset{\vdots\, P}{\vdash \Gamma} \atop {=\!=\!=\!=\!=\, \texttt{wk}} \atop {\vdash \Gamma, \Delta}$$

$$\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma, F, F}}{\vdash \Gamma, F}\,\texttt{ctr} \qquad \overset{\vdots\, Q}{\vdash \Delta, \overline{F}}}{\vdash \Gamma, \Delta}\,\texttt{cut} \quad \longrightarrow \quad \dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma, F, F} \quad \overset{\vdots\, Q}{\vdash \Delta, \overline{F}}}{\vdash \Gamma, \Delta}\,\texttt{cut} \qquad \overset{\vdots\, Q}{\vdash \Delta, \overline{F}}}{\vdash \Gamma, \Delta, \Delta}\,\texttt{cut} \atop {=\!=\!=\!=\!=\, \texttt{ctr}} \atop {\vdash \Gamma, \Delta}$$

CUT-ELIMINATION — COMMUTATIVE CASES (EXAMPLES)

$$\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma, F, G, H}}{\vdash \Gamma, F \vee G, H}\vee \qquad \overset{\vdots\, Q}{\vdash \Delta, \overline{H}}}{\vdash \Gamma, F \vee G, \Delta}\,\texttt{cut} \quad \longrightarrow \quad \dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma, F, G, H} \quad \overset{\vdots\, Q}{\vdash \Delta, \overline{H}}}{\vdash \Gamma, F, G, \Delta}\,\texttt{cut}}{\vdash \Gamma, F \vee G, \Delta}\vee$$

$$\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma_1, F, H} \quad \overset{\vdots\, Q}{\vdash \Gamma_2, G}}{\vdash \Gamma_1, \Gamma_2, F \wedge G, H}\wedge \qquad \overset{\vdots\, R}{\vdash \Delta, \overline{H}}}{\vdash \Gamma_1, \Gamma_2, F \wedge G, \Delta}\,\texttt{cut} \quad \longrightarrow \quad \dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma_1, F, H} \quad \overset{\vdots\, R}{\vdash \Delta, \overline{H}}}{\vdash \Gamma_1, F, \Delta}\,\texttt{cut} \qquad \overset{\vdots\, Q}{\vdash \Gamma_2, G}}{\vdash \Gamma_1, \Gamma_2, F \wedge G, \Delta}\wedge$$

PATHOLOGICAL CRITICAL PAIRS

$$\dfrac{\dfrac{\overset{\vdots\;P}{\vdash \Gamma,\overline{F},\overline{G}} \quad \overset{\vdots\;Q}{\vdash \Delta_1,F}}{\vdash \Gamma,\Delta_1,\overline{G}}\,\mathtt{cut} \quad \overset{\vdots\;R}{\vdash \Delta_2,G}}{\vdash \Gamma,\Delta_1,\Delta_2}\,\mathtt{cut} \quad\longleftarrow\quad \dfrac{\dfrac{\overset{\vdots\;P}{\vdash \Gamma,\overline{F},\overline{G}}}{\vdash \Gamma,\overline{F}\vee\overline{G}}\,\vee \quad \dfrac{\overset{\vdots\;Q}{\vdash \Delta_1,F} \quad \overset{\vdots\;R}{\vdash \Delta_2,G}}{\vdash \Delta_1,\Delta_2,F\wedge G}\,\wedge}{\vdash \Gamma,\Delta_1,\Delta_2}\,\mathtt{cut} \quad\longrightarrow\quad \dfrac{\dfrac{\overset{\vdots\;P}{\vdash \Gamma,\overline{F},\overline{G}} \quad \overset{\vdots\;R}{\vdash \Delta_2,G}}{\vdash \Gamma,\overline{F},\Delta_2}\,\mathtt{cut} \quad \overset{\vdots\;Q}{\vdash \Delta_1,F}}{\vdash \Gamma,\Delta_1,\Delta_2}\,\mathtt{cut}$$

$$\begin{array}{c}\vdots\;P\\ \vdash \Gamma\\ \hline\hline \vdash \Gamma,\Delta\end{array}\,\mathtt{wk} \quad\longleftarrow\quad \dfrac{\dfrac{\overset{\vdots\;P}{\vdash \Gamma}}{\vdash \Gamma,F}\,\mathtt{wk} \quad \dfrac{\overset{\vdots\;Q}{\vdash \Delta}}{\vdash \Delta,\overline{F}}\,\mathtt{wk}}{\vdash \Gamma,\Delta}\,\mathtt{cut} \quad\longrightarrow\quad \begin{array}{c}\vdots\;Q\\ \vdash \Delta\\ \hline\hline \vdash \Gamma,\Delta\end{array}\,\mathtt{wk}$$

$$\begin{array}{c}\dfrac{\dfrac{\overset{\vdots\;P}{\vdash \Gamma,F,F} \quad \dfrac{\overset{\vdots\;Q}{\vdash \Delta,\overline{F},\overline{F}}}{\vdash \Delta,\overline{F}}\,\mathtt{ctr}}{\vdash \Gamma,\Delta}\,\mathtt{cut} \quad \dfrac{\overset{\vdots\;Q}{\vdash \Delta,\overline{F},\overline{F}}}{\vdash \Delta,\overline{F}}\,\mathtt{ctr}}{\vdash \Gamma,\Delta,\Delta}\,\mathtt{cut}\\ \hline\hline \vdash \Gamma,\Delta\end{array}\,\mathtt{ctr} \quad\longleftarrow\quad \dfrac{\dfrac{\overset{\vdots\;P}{\vdash \Gamma,F,F}}{\vdash \Gamma,F}\,\mathtt{ctr} \quad \dfrac{\overset{\vdots\;Q}{\vdash \Delta,\overline{F},\overline{F}}}{\vdash \Delta,\overline{F}}\,\mathtt{ctr}}{\vdash \Gamma,\Delta}\,\mathtt{cut} \quad\longrightarrow\quad \begin{array}{c}\dfrac{\dfrac{\overset{\vdots\;P}{\vdash \Gamma,F,F}}{\vdash \Gamma,F}\,\mathtt{ctr} \quad \dfrac{\dfrac{\overset{\vdots\;P}{\vdash \Gamma,F,F}}{\vdash \Gamma,F}\,\mathtt{ctr} \quad \overset{\vdots\;Q}{\vdash \Delta,\overline{F},\overline{F}}}{\vdash \Gamma,\Delta}\,\mathtt{cut}}{\vdash \Gamma,\Gamma}\,\mathtt{cut}\\ \hline\hline \vdash \Gamma,\Delta\end{array}\,\mathtt{ctr}$$

PATHOLOGICAL CRITICAL PAIRS

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\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma, \overline{F}, \overline{G}} \quad \overset{\vdots\, Q}{\vdash \Delta_1, F}}{\vdash \Gamma, \Delta_1, \overline{G}}\,\texttt{cut} \qquad \overset{\vdots\, R}{\vdash \Delta_2, G}}{\vdash \Gamma, \Delta_1, \Delta_2}\,\texttt{cut}
\quad\longleftarrow\quad
\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma, \overline{F}, \overline{G}}}{\vdash \Gamma, \overline{F} \vee \overline{G}}\,\vee \qquad \dfrac{\overset{\vdots\, Q}{\vdash \Delta_1, F} \quad \overset{\vdots\, R}{\vdash \Delta_2, G}}{\vdash \Delta_1, \Delta_2, F \wedge G}\,\wedge}{\vdash \Gamma, \Delta_1, \Delta_2}\,\texttt{cut}
\quad\longrightarrow\quad
\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma, \overline{F}, \overline{G}} \quad \overset{\vdots\, R}{\vdash \Delta_2, G}}{\vdash \Gamma, \overline{F}, \Delta_2}\,\texttt{cut} \qquad \overset{\vdots\, Q}{\vdash \Delta_1, F}}{\vdash \Gamma, \Delta_1, \Delta_2}\,\texttt{cut}
$$

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\dfrac{\overset{\vdots\, P}{\vdash \Gamma}}{\vdash \Gamma, \Delta}\,\texttt{wk}
\quad\longleftarrow\quad
\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma}}{\vdash \Gamma, F}\,\texttt{wk} \qquad \dfrac{\overset{\vdots\, Q}{\vdash \Delta}}{\vdash \Delta, \overline{F}}\,\texttt{wk}}{\vdash \Gamma, \Delta}\,\texttt{cut}
\quad\longrightarrow\quad
\dfrac{\overset{\vdots\, Q}{\vdash \Delta}}{\vdash \Gamma, \Delta}\,\texttt{wk}
$$

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\dfrac{\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma, F, F} \quad \dfrac{\overset{\vdots\, Q}{\vdash \Delta, \overline{F}, \overline{F}}}{\vdash \Delta, \overline{F}}\,\texttt{ctr}}{\vdash \Gamma, \Delta}\,\texttt{cut} \qquad \dfrac{\overset{\vdots\, Q}{\vdash \Delta, \overline{F}, \overline{F}}}{\vdash \Delta, \overline{F}}\,\texttt{ctr}}{\vdash \Gamma, \Delta, \Delta}\,\texttt{cut}}{\vdash \Gamma, \Delta}\,\texttt{ctr}
\quad\longleftarrow\quad
\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma, F, F}}{\vdash \Gamma, F}\,\texttt{ctr} \qquad \dfrac{\overset{\vdots\, Q}{\vdash \Delta, \overline{F}, \overline{F}}}{\vdash \Delta, \overline{F}}\,\texttt{ctr}}{\vdash \Gamma, \Delta}\,\texttt{cut}
\quad\longrightarrow\quad
\dfrac{\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma, F, F}}{\vdash \Gamma, F}\,\texttt{ctr} \qquad \dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma, F, F}}{\vdash \Gamma, F}\,\texttt{ctr} \quad \overset{\vdots\, Q}{\vdash \Delta, \overline{F}, \overline{F}}}{\vdash \Gamma, \Delta}\,\texttt{cut}}{\vdash \Gamma, \Gamma}\,\texttt{cut}}{\vdash \Gamma, \Delta}\,\texttt{ctr}
$$

LAFONT'S EXAMPLE & CO. (PROOFS AND TYPES, 1989)

$$\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma}}{\vdash \Gamma, \Gamma}\,\texttt{wk}}{\vdash \Gamma}\,\texttt{ctr} = P' \quad \longleftarrow \quad \dfrac{\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma}}{\vdash \Gamma, F}\,\texttt{wk} \qquad \dfrac{\overset{\vdots\, Q}{\vdash \Gamma}}{\vdash \Gamma, \overline{F}}\,\texttt{wk}}{\vdash \Gamma, \Gamma}\,\texttt{cut}}{\vdash \Gamma}\,\texttt{ctr} \quad \longrightarrow \quad Q' = \dfrac{\dfrac{\overset{\vdots\, Q}{\vdash \Gamma}}{\vdash \Gamma, \Gamma}\,\texttt{wk}}{\vdash \Gamma}\,\texttt{ctr}$$

Assumption 1. *There is a function* $[\![-]\!] : \texttt{LK} \to X$ *mapping LK derivations to interpretations in some space* X.

Assumption 2. *The interpretation is preserved under cut reduction, i.e. for all* $R, R' \in \texttt{LK}$, *if* $R \longrightarrow R'$ *then* $[\![R]\!] = [\![R']\!]$.

Assumption 3. *The interpretation is such that* $[\![P']\!] = [\![P]\!]$ *and* $[\![Q']\!] = [\![Q]\!]$.

Corollary. $[\![P']\!] = [\![Q']\!]$, *hence* $[\![P]\!] = [\![Q]\!]$.

LAFONT'S EXAMPLE & CO. (PROOFS AND TYPES, 1989)

$$
\dfrac{\dfrac{\begin{matrix}\vdots\, P \\ \vdash F\end{matrix}}{\vdash F, G}\,\mathtt{wk}}{\vdash F \vee G}\,\vee \;=\; P'
\quad\longleftarrow\quad
\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\, P \\ \vdash F\end{matrix}}{\vdash F, H}\,\mathtt{wk} \qquad \dfrac{\begin{matrix}\vdots\, Q \\ \vdash G\end{matrix}}{\vdash G, \overline{H}}\,\mathtt{wk}}{\vdash F, G}\,\mathtt{cut}}{\vdash F \vee G}\,\vee
\quad\longrightarrow\quad
Q' \;=\; \dfrac{\dfrac{\begin{matrix}\vdots\, Q \\ \vdash G\end{matrix}}{\vdash F, G}\,\mathtt{wk}}{\vdash F \vee G}\,\vee
$$

Assumption 1. *There is a function* $[\![-]\!] : \mathtt{LK} \to X$ *mapping LK derivations to interpretations in some space* X*.*

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Corollary. $[\![P']\!] = [\![Q']\!]$.

LAFONT'S EXAMPLE & CO. (PROOFS AND TYPES, 1989)

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\dfrac{\begin{matrix}\vdots\, P \\ \vdash \Gamma\end{matrix} \qquad \begin{matrix}\vdots\, Q \\ \vdash \Delta\end{matrix}}{\vdash \Gamma, \Delta}\,\texttt{mix}
\quad \overset{?}{\longleftarrow} \quad
\dfrac{\dfrac{\begin{matrix}\vdots\, P \\ \vdash \Gamma\end{matrix}}{\vdash \Gamma, F}\,\texttt{wk} \qquad \dfrac{\begin{matrix}\vdots\, Q \\ \vdash \Delta\end{matrix}}{\vdash \Delta, \overline{F}}\,\texttt{wk}}{\vdash \Gamma, \Delta}\,\texttt{cut}
\quad \overset{?}{\longrightarrow} \quad
\dfrac{\dfrac{\begin{matrix}\vdots\, P \\ \vdash \Gamma\end{matrix}}{\vdash \Gamma, \Delta}\,\texttt{wk} \qquad \dfrac{\begin{matrix}\vdots\, Q \\ \vdash \Delta\end{matrix}}{\vdash \Gamma, \Delta}\,\texttt{wk}}{\vdash \Gamma, \Delta}\,\oplus
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PATHOLOGICAL CRITICAL PAIRS

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\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma, \overline{F}, \overline{G}} \quad \overset{\vdots\, Q}{\vdash \Delta_1, F}}{\vdash \Gamma, \Delta_1, \overline{G}}\,\texttt{cut} \qquad \overset{\vdots\, R}{\vdash \Delta_2, G}}{\vdash \Gamma, \Delta_1, \Delta_2}\,\texttt{cut}
\quad\longleftarrow\quad
\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma, \overline{F}, \overline{G}}}{\vdash \Gamma, \overline{F} \vee \overline{G}}\,\vee \qquad \dfrac{\overset{\vdots\, Q}{\vdash \Delta_1, F} \quad \overset{\vdots\, R}{\vdash \Delta_2, G}}{\vdash \Delta_1, \Delta_2, F \wedge G}\,\wedge}{\vdash \Gamma, \Delta_1, \Delta_2}\,\texttt{cut}
\quad\longrightarrow\quad
\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma, \overline{F}, \overline{G}} \quad \overset{\vdots\, R}{\vdash \Delta_2, G}}{\vdash \Gamma, \overline{F}, \Delta_2}\,\texttt{cut} \qquad \overset{\vdots\, Q}{\vdash \Delta_1, F}}{\vdash \Gamma, \Delta_1, \Delta_2}\,\texttt{cut}
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\dfrac{\overset{\vdots\, P}{\vdash \Gamma}}{\vdash \Gamma, \Delta}\,\texttt{wk}
\quad\longleftarrow\quad
\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma}}{\vdash \Gamma, F}\,\texttt{wk} \qquad \dfrac{\overset{\vdots\, Q}{\vdash \Delta}}{\vdash \Delta, \overline{F}}\,\texttt{wk}}{\vdash \Gamma, \Delta}\,\texttt{cut}
\quad\longrightarrow\quad
\dfrac{\overset{\vdots\, Q}{\vdash \Delta}}{\vdash \Gamma, \Delta}\,\texttt{wk}
$$

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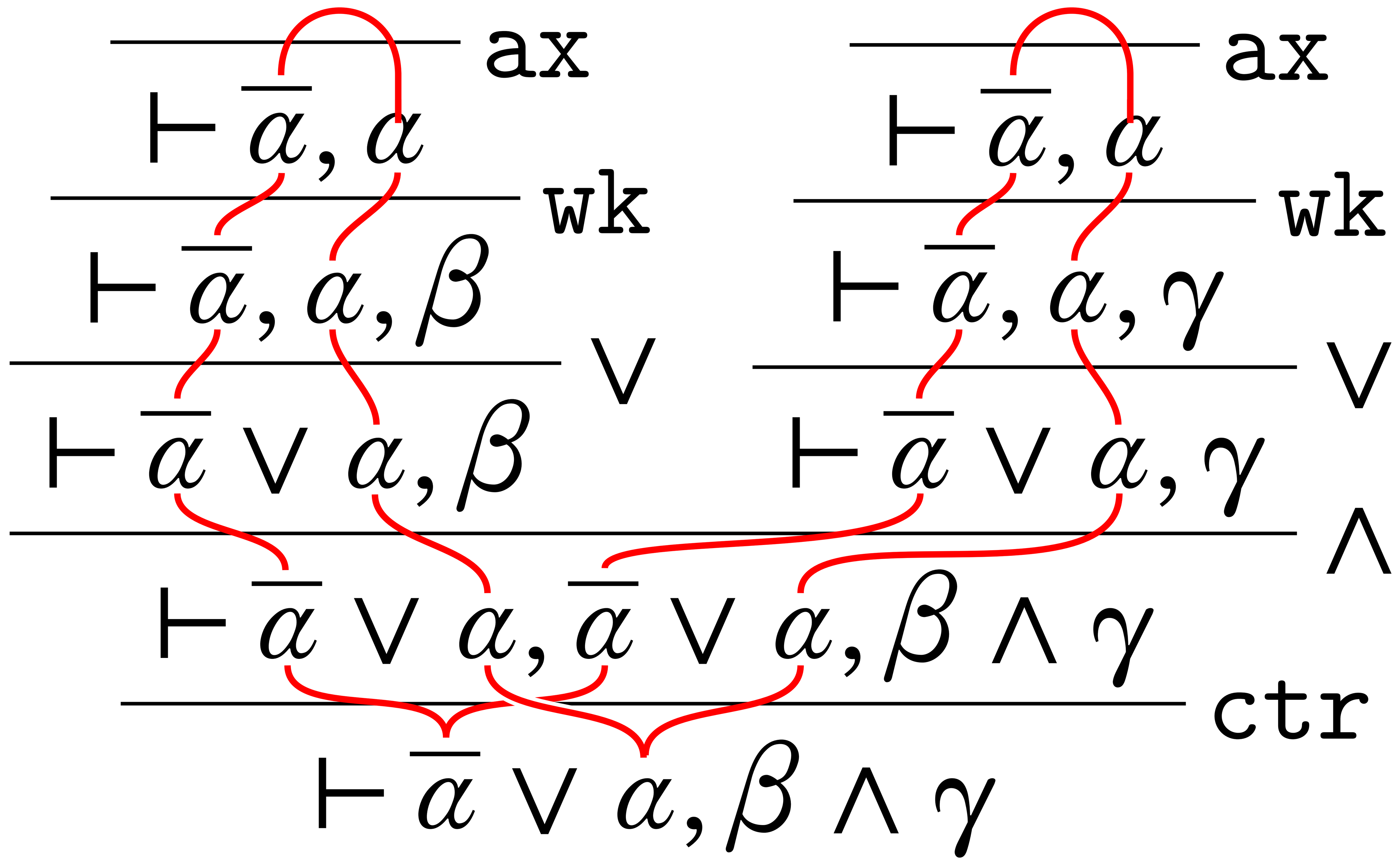
\dfrac{\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma, F, F} \quad \dfrac{\overset{\vdots\, Q}{\vdash \Delta, \overline{F}, \overline{F}}}{\vdash \Delta, \overline{F}}\,\texttt{ctr}}{\vdash \Gamma, \Delta}\,\texttt{cut} \qquad \dfrac{\overset{\vdots\, Q}{\vdash \Delta, \overline{F}, \overline{F}}}{\vdash \Delta, \overline{F}}\,\texttt{ctr}}{\vdash \Gamma, \Delta, \Delta}\,\texttt{cut}}{\vdash \Gamma, \Delta}\,\texttt{ctr}
\quad\longleftarrow\quad
\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma, F, F}}{\vdash \Gamma, F}\,\texttt{ctr} \qquad \dfrac{\overset{\vdots\, Q}{\vdash \Delta, \overline{F}, \overline{F}}}{\vdash \Delta, \overline{F}}\,\texttt{ctr}}{\vdash \Gamma, \Delta}\,\texttt{cut}
\quad\longrightarrow\quad
\dfrac{\dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma, F, F}}{\vdash \Gamma, F}\,\texttt{ctr} \qquad \dfrac{\dfrac{\overset{\vdots\, P}{\vdash \Gamma, F, F}}{\vdash \Gamma, F}\,\texttt{ctr} \quad \overset{\vdots\, Q}{\vdash \Delta, \overline{F}, \overline{F}}}{\vdash \Gamma, \Delta}\,\texttt{cut}}{\vdash \Gamma, \Gamma}\,\texttt{cut}}{\vdash \Gamma, \Delta}\,\texttt{ctr}
$$

IDEA #1 — TRACKING AXIOMS

Track existence, avoid counting.
(Andrews 1976, Lamarche & Straßburger 2004+)



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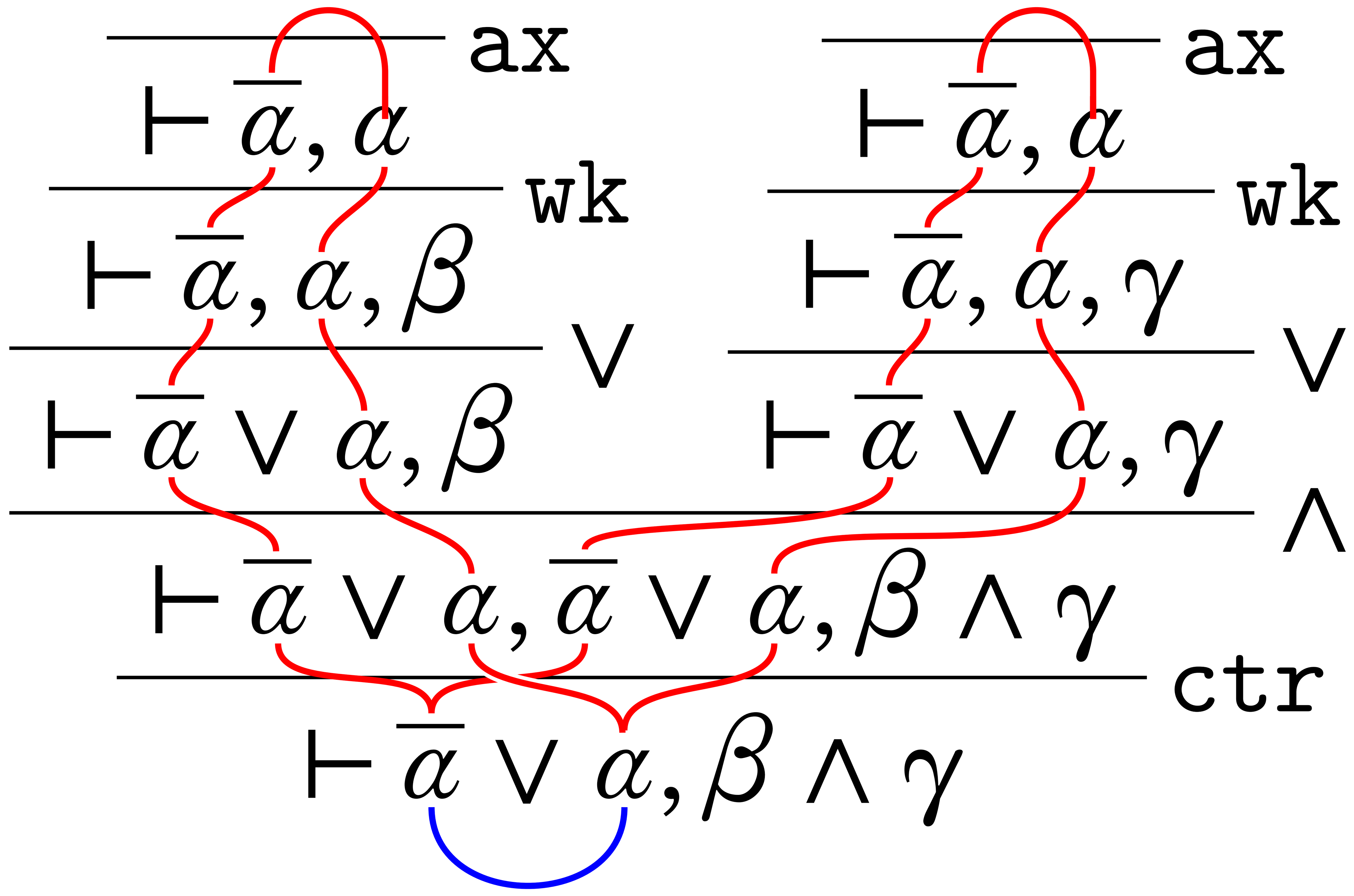

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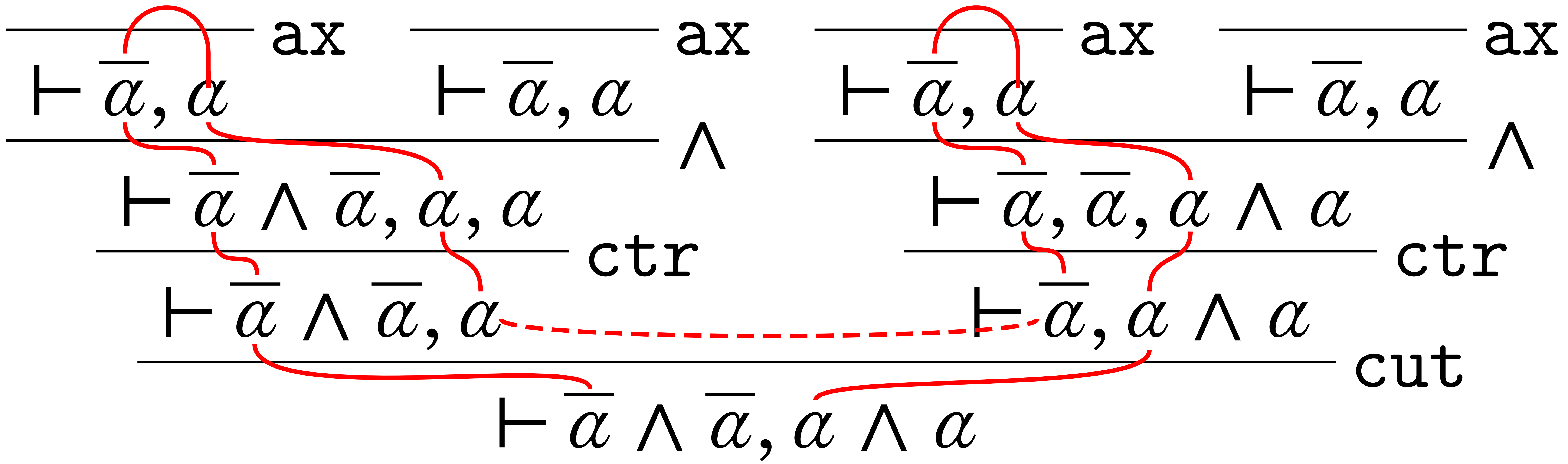

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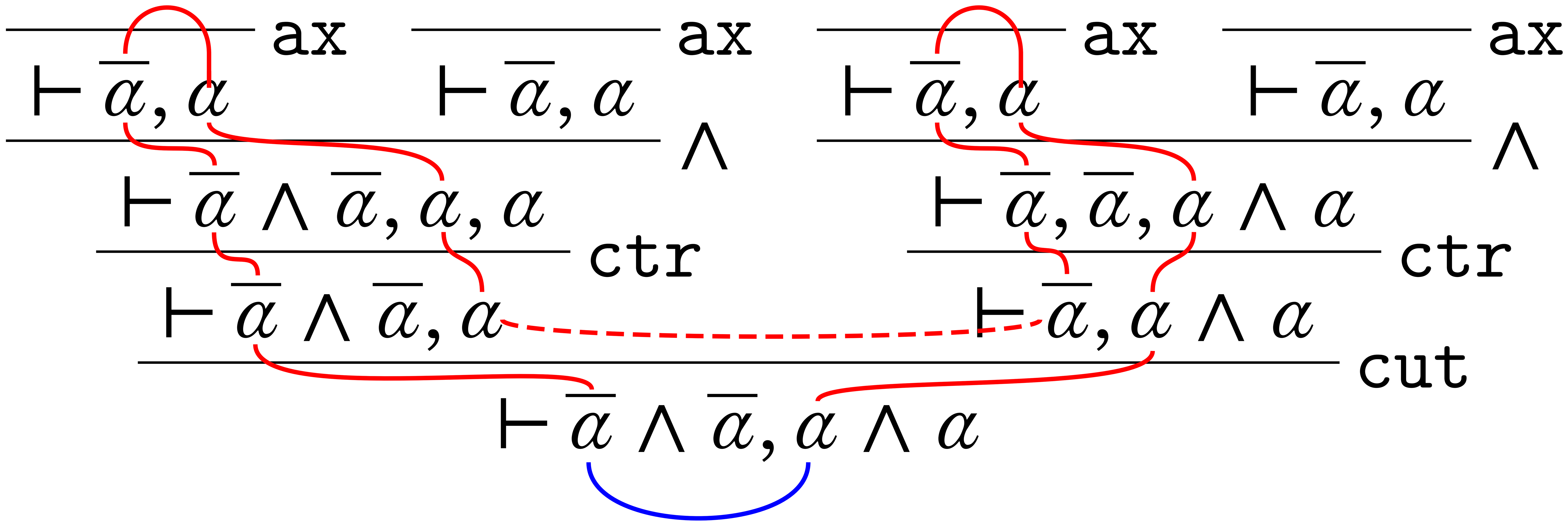

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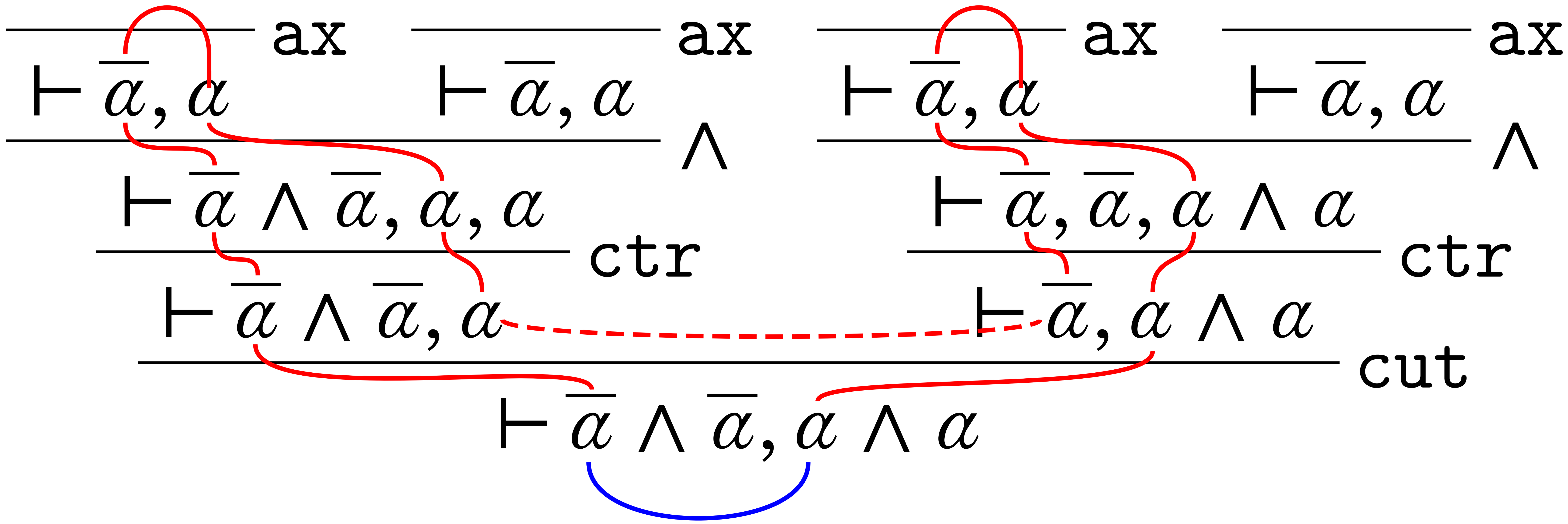
Theorem (Führmann & Pym 2004). *Mating graphs decrease under cut-reduction.*



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Axioms may be preserved, deleted,
duplicated, but *never created*.

$$\dfrac{\dfrac{\dfrac{}{\vdash \overline{F}, F}\,\texttt{ax} \qquad \dfrac{\begin{array}{c}\vdots\\ \vdash G, H, H\end{array}}{\vdash G, H}\,\texttt{ctr}}{\vdash \overline{F}, F \wedge G, H}\,\wedge \qquad \dfrac{\begin{array}{c}\vdots\\ \vdash \overline{H}, \overline{H}\end{array}}{\vdash \overline{H}}\,\texttt{ctr}}{\vdash \overline{F}, F \wedge G}\,\texttt{cut}$$

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$$\dfrac{\dfrac{}{\vdash \overline{F}, F}\,\texttt{ax} \qquad \dfrac{\dfrac{\genfrac{}{}{0pt}{}{\vdots}{\vdash G, H, H}}{\vdash G, H}\,\texttt{ctr} \qquad \dfrac{\genfrac{}{}{0pt}{}{\vdots}{\vdash \overline{H}, \overline{H}}}{\vdash \overline{H}}\,\texttt{ctr}}{\vdash G}\,\texttt{cut}}{\vdash \overline{F}, F \wedge G}\,\wedge$$

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  \dfrac{
    \dfrac{}{\vdash \overline{F}, F}\,\texttt{ax}
    \qquad
    \dfrac{\stackrel{\vdots}{\vdash G, H, H}}{\vdash G, H}\,\texttt{ctr}
  }{\vdash \overline{F}, F \wedge G, H}\,\wedge
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{}{\vdash \overline{F}, F}\,\texttt{ax}
      \qquad
      \dfrac{\stackrel{\vdots}{\vdash G, H, H}}{\vdash G, H}\,\texttt{ctr}
    }{\vdash \overline{F}, F \wedge G, H}\,\wedge
    \qquad
    \dfrac{\stackrel{\vdots}{\vdash \overline{H}, \overline{H}}}{\vdash \overline{H}}\,\texttt{ctr}
  }{\vdash \overline{F}, F \wedge G, \overline{H}}\,\texttt{cut}
}{
  \dfrac{\vdash \overline{F}, F \wedge G, \overline{F}, F \wedge G}{\vdash \overline{F}, F \wedge G}\,\texttt{ctr}
}\,\texttt{cut}
$$

MATINGS ARE NOT INVARIANTS!

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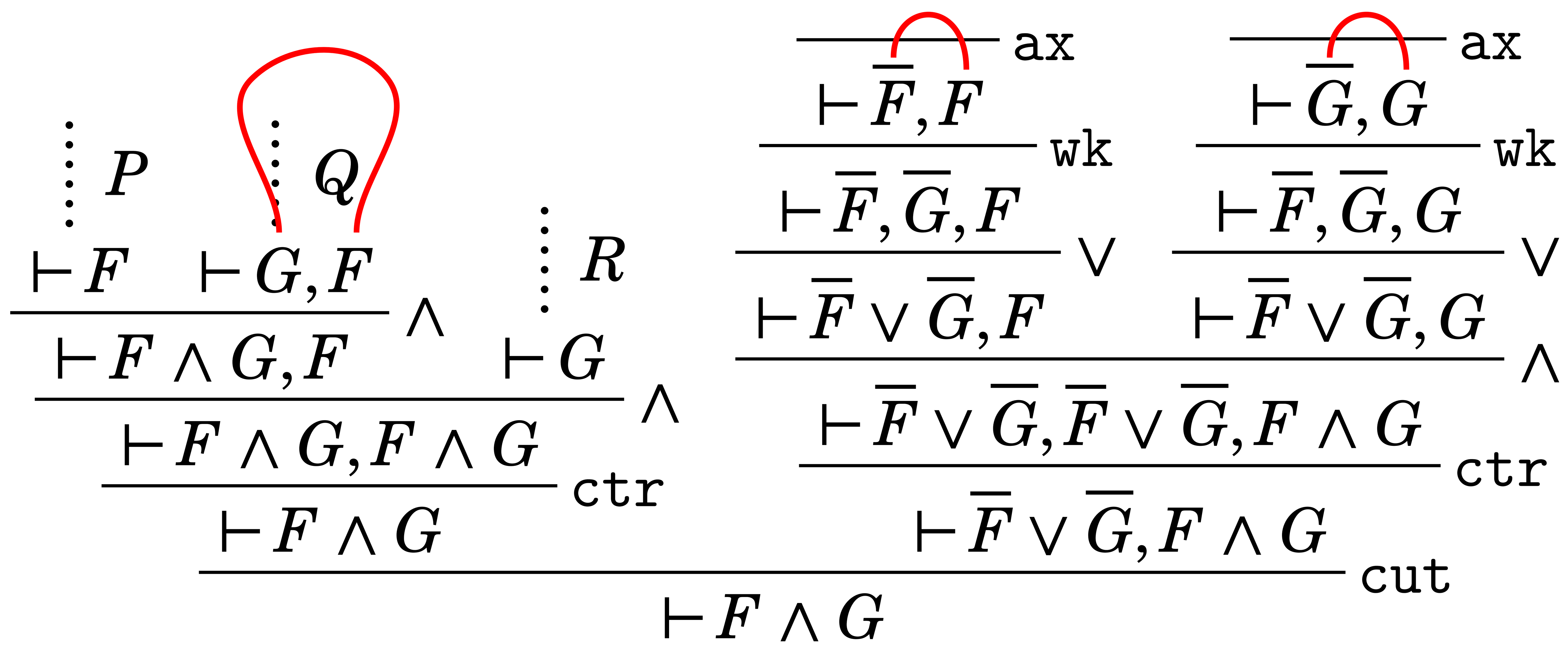
\overset{\vdots\; P}{\vdash F} \qquad \overset{\vdots\; Q}{\vdash G, F}
}{\vdash F \wedge G, F}\,\wedge
\qquad \overset{\vdots\; R}{\vdash G}
}{\vdash F \wedge G, F \wedge G}\,\wedge
}{\vdash F \wedge G}\,\texttt{ctr}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{\vdash \overline{F}, F}\,\texttt{ax}
}{\vdash \overline{F}, \overline{G}, F}\,\texttt{wk}
}{\vdash \overline{F} \vee \overline{G}, F}\,\vee
\qquad
\dfrac{
\dfrac{
\dfrac{}{\vdash \overline{G}, G}\,\texttt{ax}
}{\vdash \overline{F}, \overline{G}, G}\,\texttt{wk}
}{\vdash \overline{F} \vee \overline{G}, G}\,\vee
}{\vdash \overline{F} \vee \overline{G}, \overline{F} \vee \overline{G}, F \wedge G}\,\wedge
}{\vdash \overline{F} \vee \overline{G}, F \wedge G}\,\texttt{ctr}
}{\vdash F \wedge G}\,\texttt{cut}
$$

MATINGS ARE NOT INVARIANTS!



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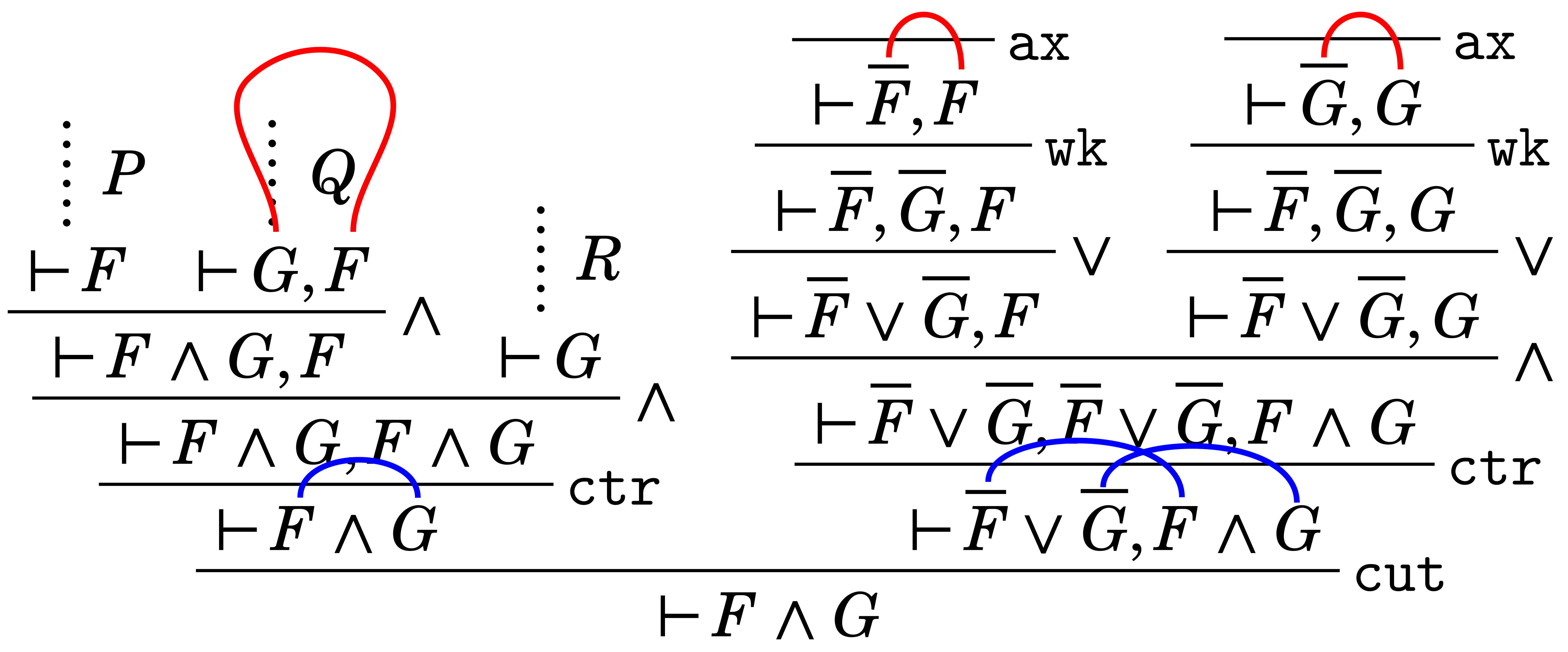

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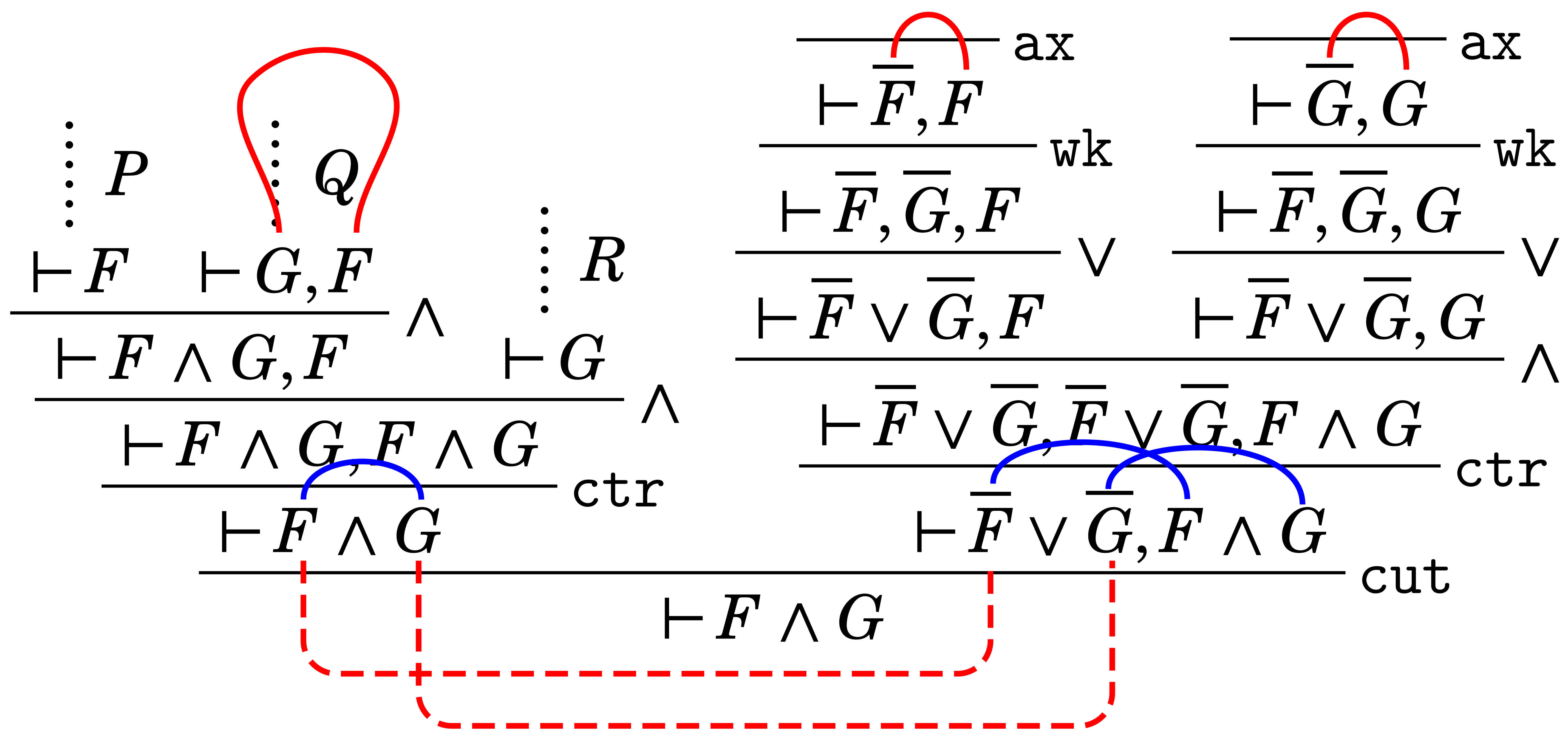

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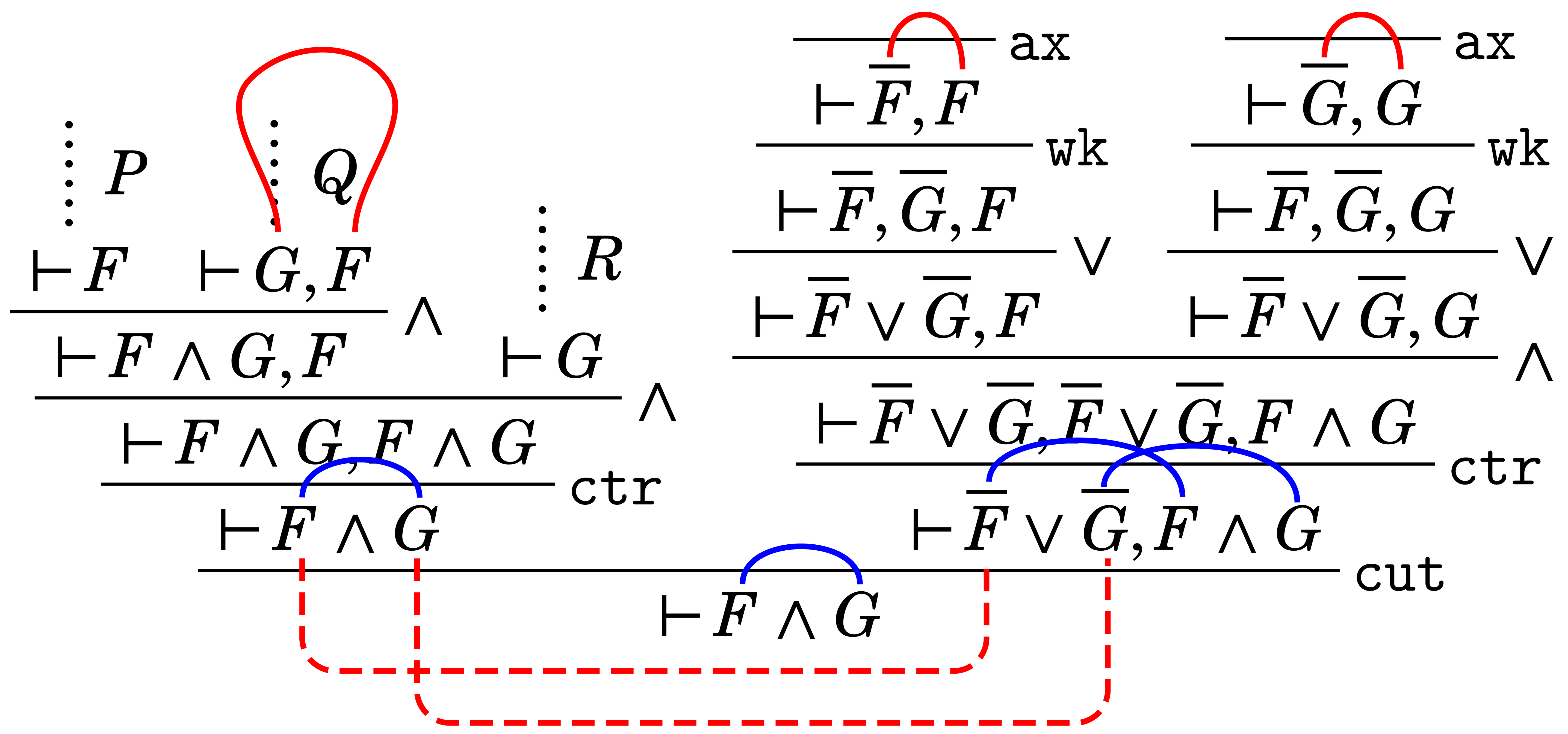

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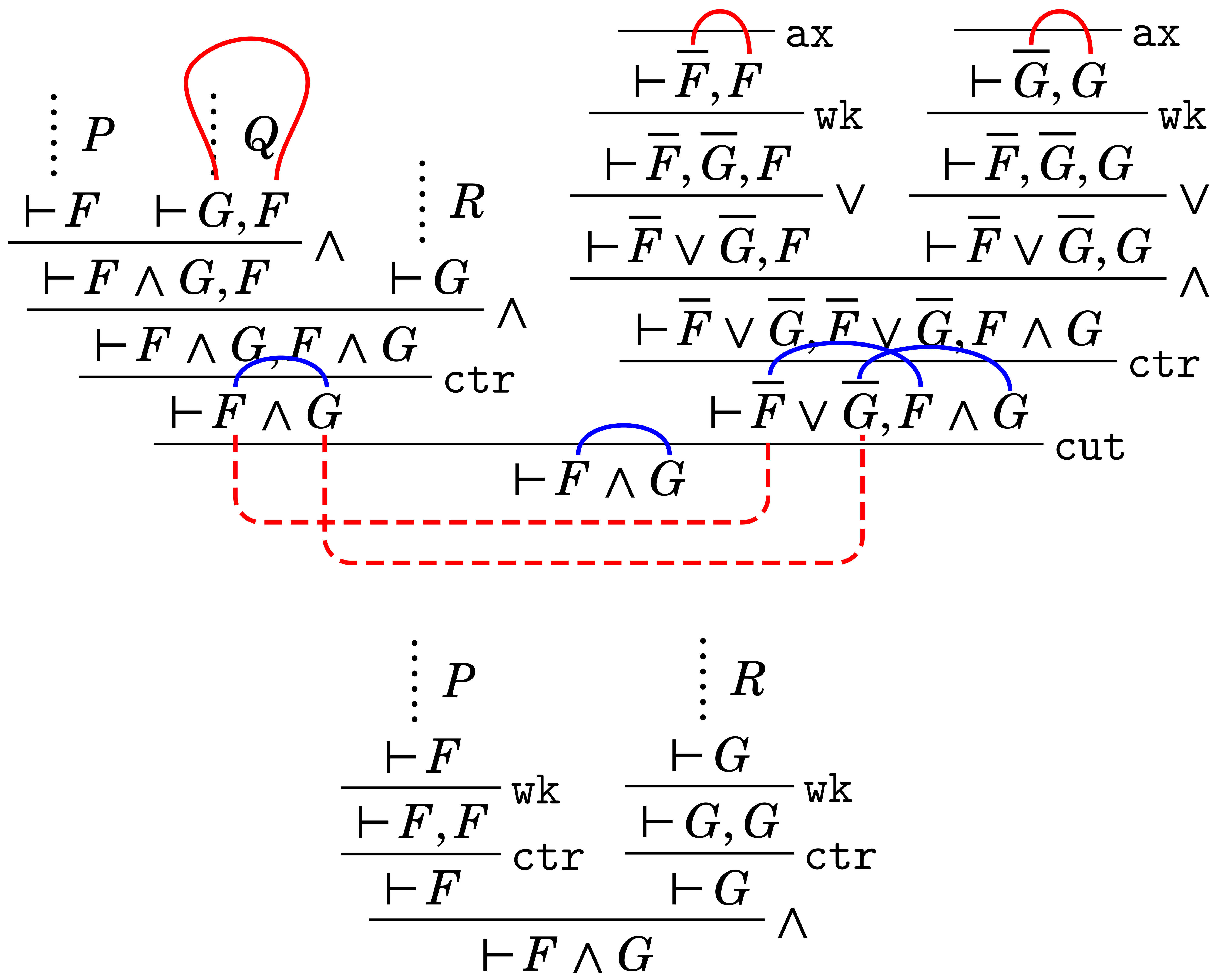


IDEA #2 — TARGET INVERTIBILITY

Identity group:

$$\frac{}{\vdash \Gamma, A, \overline{A}}\,\texttt{ax} \qquad \frac{\vdash \Gamma, A \quad \vdash \Gamma, \overline{A}}{\vdash \Gamma}\,\texttt{cut}$$

Structural group:

$$\frac{\vdash \Gamma \quad \vdash \Gamma}{\vdash \Gamma}\,\texttt{sum}$$

Logical group:

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}\,\vee \qquad \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\,\wedge$$

IDEA #2 — TARGET INVERTIBILITY

$$\dfrac{\dfrac{\vdash A, B, C \quad \vdash A, B, D}{\vdash A, B, C \wedge D}\wedge}{\vdash A \vee B, C \wedge D}\vee \quad \longleftrightarrow \quad \dfrac{\dfrac{\vdash A, B, C}{\vdash A \vee B, C}\vee \quad \dfrac{\vdash A, B, D}{\vdash A \vee B, D}\vee}{\vdash A \vee B, C \wedge D}\wedge$$

$$\dfrac{\dfrac{\vdash A, C \quad \vdash A, D}{\vdash A, C \wedge D}\wedge \quad \dfrac{\vdash B, C \quad \vdash B, D}{\vdash B, C \wedge D}\wedge}{\vdash A \wedge B, C \wedge D}\wedge \quad \longleftrightarrow \quad \dfrac{\dfrac{\vdash A, C \quad \vdash B, C}{\vdash A \wedge B, C}\wedge \quad \dfrac{\vdash A, D \quad \vdash B, D}{\vdash A \wedge B, D}\wedge}{\vdash A \wedge B, C \wedge D}\wedge$$

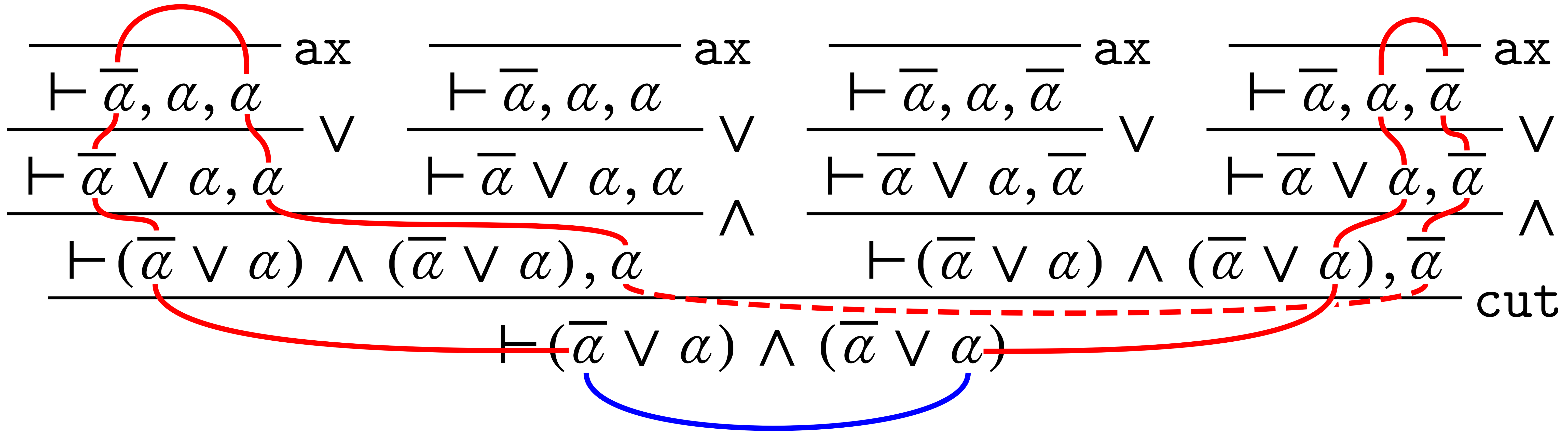
IDEA #2 — TARGET INVERTIBILITY

$$\dfrac{\dfrac{\vdash A,B,C \quad \vdash A,B,D}{\vdash A,B,C\land D}\land}{\vdash A\lor B,C\land D}\lor \qquad \longleftrightarrow \qquad \dfrac{\dfrac{\vdash A,B,C}{\vdash A\lor B,C}\lor \quad \dfrac{\vdash A,B,D}{\vdash A\lor B,D}\lor}{\vdash A\lor B,C\land D}\land$$

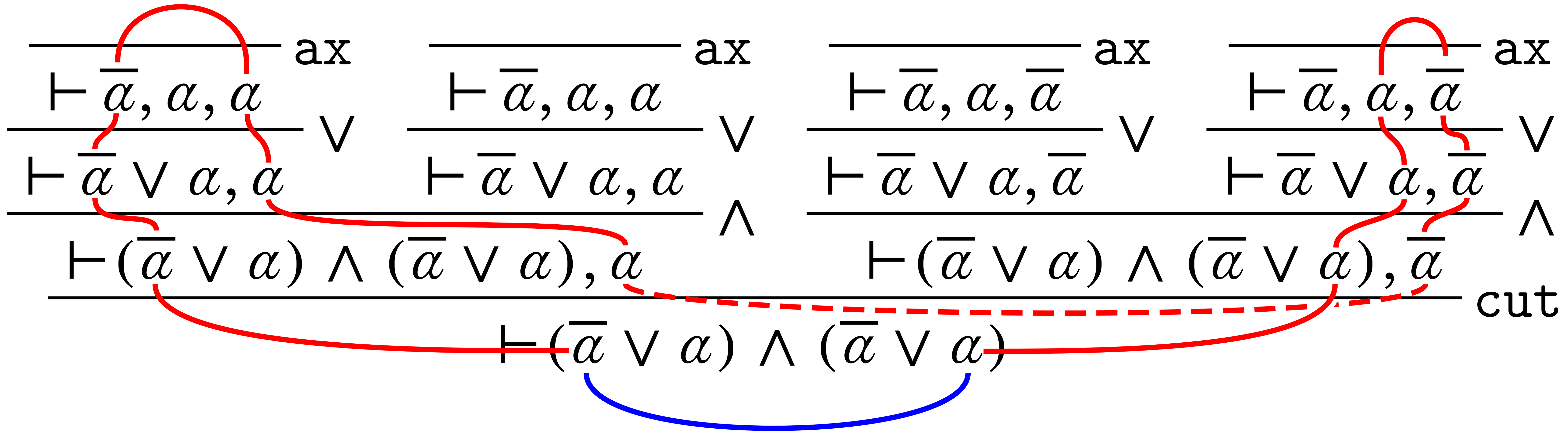
$$\dfrac{\dfrac{\vdash A,C \quad \vdash A,D}{\vdash A,C\land D}\land \quad \dfrac{\vdash B,C \quad \vdash B,D}{\vdash B,C\land D}\land}{\vdash A\land B,C\land D}\land \qquad \longleftrightarrow \qquad \dfrac{\dfrac{\vdash A,C \quad \vdash B,C}{\vdash A\land B,C}\land \quad \dfrac{\vdash A,D \quad \vdash B,D}{\vdash A\land B,D}\land}{\vdash A\land B,C\land D}\land$$

Theorem. *Mating graphs are invariant under arbitrary permutations of logical rules in the cut-free fragment of GS3.*

IDEA #3 — ADD BRANCH ANNOTATIONS

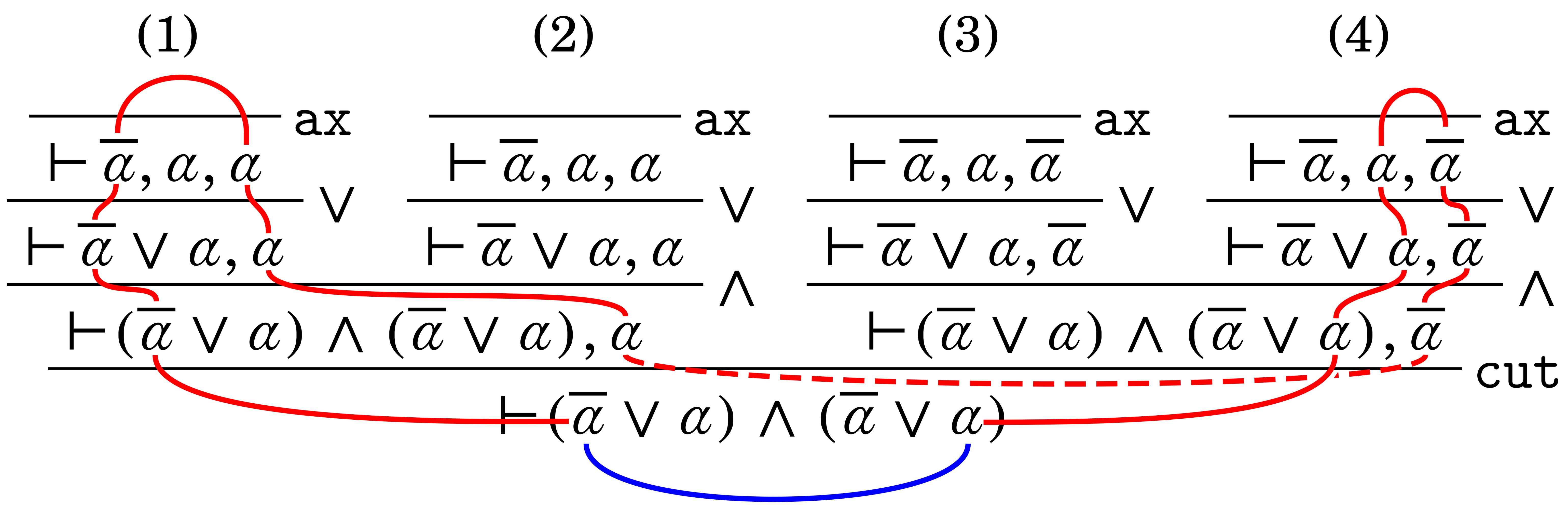


IDEA #3 — ADD BRANCH ANNOTATIONS



$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \overline{\alpha}, \alpha, \alpha}\,\mathtt{ax}}{\vdash \overline{\alpha} \vee \alpha, \alpha}\,\vee \quad \dfrac{\dfrac{}{\vdash \overline{\alpha}, \alpha, \overline{\alpha}}\,\mathtt{ax}}{\vdash \overline{\alpha} \vee \alpha, \overline{\alpha}}\,\vee}{\vdash \overline{\alpha} \vee \alpha}\,\mathtt{cut} \qquad \dfrac{\dfrac{\dfrac{}{\vdash \overline{\alpha}, \alpha, \alpha}\,\mathtt{ax}}{\vdash \overline{\alpha} \vee \alpha, \alpha}\,\vee \quad \dfrac{\dfrac{}{\vdash \overline{\alpha}, \alpha, \overline{\alpha}}\,\mathtt{ax}}{\vdash \overline{\alpha} \vee \alpha, \overline{\alpha}}\,\vee}{\vdash \overline{\alpha} \vee \alpha}\,\mathtt{cut}}{\vdash (\overline{\alpha} \vee \alpha) \wedge (\overline{\alpha} \vee \alpha)}\,\wedge$$

IDEA #3 — ADD BRANCH ANNOTATIONS



$$\dfrac{\dfrac{\overset{(1)}{\dfrac{\dfrac{}{\vdash \overline{\alpha}, \alpha, \alpha}\,\mathtt{ax}}{\vdash \overline{\alpha} \vee \alpha, \alpha}\,\vee} \qquad \overset{(3)}{\dfrac{\dfrac{}{\vdash \overline{\alpha}, \alpha, \overline{\alpha}}\,\mathtt{ax}}{\vdash \overline{\alpha} \vee \alpha, \overline{\alpha}}\,\vee}}{\vdash \overline{\alpha} \vee \alpha}\,\mathtt{cut} \qquad \dfrac{\overset{(2)}{\dfrac{\dfrac{}{\vdash \overline{\alpha}, \alpha, \alpha}\,\mathtt{ax}}{\vdash \overline{\alpha} \vee \alpha, \alpha}\,\vee} \qquad \overset{(4)}{\dfrac{\dfrac{}{\vdash \overline{\alpha}, \alpha, \overline{\alpha}}\,\mathtt{ax}}{\vdash \overline{\alpha} \vee \alpha, \overline{\alpha}}\,\vee}}{\vdash \overline{\alpha} \vee \alpha}\,\mathtt{cut}}{\vdash (\overline{\alpha} \vee \alpha) \wedge (\overline{\alpha} \vee \alpha)}\,\wedge$$

MAIN RESULTS

Theorem (Invariance). *Branch-annotated mating graphs are invariant under arbitrary permutations of logical rules in the full calculus GS3.*

Proof sketch

1. Assign unique names to atomic formula occurrences.

2. Define branch-annotated matings as weighted graphs.

3. Define branch-sensitive composition.

4. Show that commutations between cuts and logical rules preserve all valid alternating paths.

MAIN RESULTS

Theorem (Cut-elimination). *For any GS3 derivation there is a cut-free GS3 derivation with the same conclusion and denotation.*

Proof sketch

1. Upon finding a cut, normalize recursively its sub-derivations.

2. Use logical rule permutations to reduce the conclusion of the cut to an atomic clause.

3. Compute the denotation and reconstruct the resulting derivation.

MAIN RESULTS

Theorem (Sequentialization). *A branch-annotated mating G is correct w.r.t. some sequent $\vdash \Gamma$ if and only if there is a GS3 derivation $P \vdash \Gamma$ whose denotation is G*

Moreover, the size of P is polynomially bounded by the size of G.

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Corollary. *Branch-annotated matings form a proof-system in the sense of Cook & Reckhow (1979).*

Rule permutations are identities

Efficient invertibility

Admissibility of rules by algebraic reasoning

SHORTCOMINGS

- Cuts do not commute!
- No local cut-reduction procedure.
- Has the original problem been solved? Unclear.
- No predicate calculus.

THANK YOU